# A TEST TO ASSESS THE DYNAMIC EVOLUTION OF PREFERENCES IN MARKETING SURVEYS

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***Abstract.*** *The subject of this paperis a two-stage hypothesis test, which may have interesting applications in several situations related to the opinion research field. Such test is based on the components of a Bivariate Correlated Normal random variable. In particular, it is based on the exact distribution of their minimum and maximum modulus. This test was proposed for the first time by Duncan in Miller (1981), and was recently improved by Pollastri (2008). In the latter paper a variant of such test is described, since two samples in two different times or situations are considered. Two kinds of applications are provided to show the wide range of usage.*

***Keywords:*** *Trinomial distribution, Bivariate Correlated Normal Distribution, Two-stage hypothesis test, Duncan-Pollastri test.*

## 1. INTRODUCTION

In many cases situations must be faced in which it is necessary to analyse experiments consisting in *n* independent trials and each trial has *k* possible categories. The multinomial distribution that can describe this kind of experiment has been widely studied and detailed treatment can be found in Johnson et al. (1997). Inference about the probabilities of two trinomial distribution with the preferences for three outcomes will be considered in this paper. The situation analysed can occur whenever it is necessary to study opinions regarding a social problem, the preferences for a candidate A or a candidate B, or not expressed, or for two products and a third category with all the others less important products. In particular, two samples are observed in two different situations, the purpose being to compare the probabilities of two trinomial distributions in order to verify whether the

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probabilities have changed.

Suppose that each of the independent identical trials has outcomes in three mutually exclusive categories and that the trials are repeated after some event or in a different situation or time. Hypothesis tests are commonly used to appraise evidence for consistent differences among groups. Here the question is to verify whether the proportions of the two principal categories have changed (or not) over time or in two different places.

An alternative approach is to give the confidence interval as a measure of effective size, for example between the differences of proportions. Many scholars (for example Agresti (2002), and Newcombe (2012)) declare that, in practice, it is more informative to construct confidence intervals for parameters than to test hypotheses about their values. Often inference regarding multinomial probabilities is based on Pearson's Chi-squared statistic with some improvements. Many examples of confidence regions based on Chi-squared statistics (Bailey, 1980; Goodman, 1965; Quesenberry and Hurst, 1964), or on approximate parametric bootstrap (Glaz and Sison, 1999) can be found in the literature.

The problem is that the confidence region cannot give the possibility of accepting the hypotheses connected with the increase or decrease of the probabilities, but only the null hypotheses regarding the invariance of the two principal probabilities.

The procedure proposed here is based on a two-stage test on the means of a Bivariate Correlated Normal (B.C.N.) random variable (rv) reported in Miller (1981) attributed to Duncan. Duncan's original study has never been found. The first stage of the test is based on the Bonferroni inequality, while the second is based on a univariate normal rv. Using the distribution of the absolute maximum of two correlated normal rvs due to Zenga (1979) and the absolute minimum of two correlated normal rvs due to Pollastri and Tornaghi (2004), in 2008 Pollastri proposed an improved procedure for testing the means of a B.C.N. rv. In 2012 Pollastri and Riva used this improved procedure to test the hypothesis that the probabilities of a trinomial distribution are equal to fixed values when the sample size is large. Mazurek and Ostasiewicz (2013) studied the sample size for a fixed probability of type I error using the exact distribution of all possible outcomes.

The purpose of this paper is to compare the probabilities of the two most important categories in two different situations. It is important to underline that the two-stage test accepts one of nine hypotheses, so it is possible to arrive at a decision about the modification of the means or probabilities. It is also worth remarking that the computation is simple because the critical values needed in the two stages of the procedure are reported in Pollastri (2008) and in Pollastri and Riva (2012).

This paper is organised as follows. Duncan's procedure and its improvement are presented in Section 2. Section 3 is devoted to extension of the test to the two-sample problem. Section 4 reports applications about the US Presidential election in 2012 in two different times and comparison of the level of trust in the national statistics in some European countries. Finally, Section 5 is devoted to the conclusions.

## 2. GENESIS OF THE TEST

### 2.1. DESCRIPTION OF THE DUNCAN TEST

Duncan in Miller (1981) proposed a procedure in order to test the hypothesis on the means of a B.C.N. rv. This test is a variant of the maximum modulus test. Basically it is described by the following procedure.

Let $(y_{1i}, y_{2i})$ with $(i = 1, 2, ..., n)$ be a simple random sample from a B.C.N. rv $\mathbf{Y} = (Y_1, Y_2)$ such that

$$(Y_1, Y_2) \sim N\left[(\mu_1, \mu_2); \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right].$$

Under the assumption that the marginal variances (that is the variances of the components $Y_1$ and $Y_2$ denoted by $\sigma_1^2$ and $\sigma_2^2$, respectively) are known, the Duncan test allows to infer information on the means $(\mu_1, \mu_2)$. More in detail, the test verifies the null hypothesis:

$$H_0\colon (\mu_1 = \mu_1^*) \cap (\mu_2 = \mu_2^*)$$

against all the eight possible alternatives:

$$\begin{array}{ll}
1) & (\mu_1 = \mu_1^*) \cap (\mu_2 > \mu_2^*) \\
2) & (\mu_1 = \mu_1^*) \cap (\mu_2 < \mu_2^*) \\
3) & (\mu_1 > \mu_1^*) \cap (\mu_2 > \mu_2^*) \\
4) & (\mu_1 > \mu_1^*) \cap (\mu_2 < \mu_2^*) \\
5) & (\mu_1 > \mu_1^*) \cap (\mu_2 = \mu_2^*) \\
6) & (\mu_1 < \mu_1^*) \cap (\mu_2 = \mu_2^*) \\
7) & (\mu_1 < \mu_1^*) \cap (\mu_2 > \mu_2^*) \\
8) & (\mu_1 < \mu_1^*) \cap (\mu_2 < \mu_2^*)
\end{array}$$

where the quantities $\mu_1^*$ and $\mu_2^*$ are fixed values.

The test statistic is based on the bivariate rv $(X_1, X_2)$, defined by

$$X_1 = \frac{\bar{Y}_1 - \mu_1^*}{\sigma_1/\sqrt{n}} \quad \text{and} \quad X_2 = \frac{\bar{Y}_2 - \mu_2^*}{\sigma_2/\sqrt{n}}$$

where $\bar{Y}_1$ and $\bar{Y}_2$ denote the sample means of $Y_1$ and $Y_2$, respectively. Under the null hypothesis $H_0$, $(X_1, X_2)$ has the Standard Bivariate Correlated Normal (S.B.C.N.) distribution with correlation coefficient $\rho$:

$$(X_1, X_2) \sim N\left[(0,0)\,;\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right]. \tag{1}$$

The procedure consists of two stages.

*Stage1:* for a fixed probability of type I error $\alpha$, the critical value $c$ is calculated, assuming that $|X_1|$ and $|X_2|$ are independent. The value $c$ is such that:

$$P[\max\{|X_1|, |X_2|\} \le c] = P[|X_1| \le c] \cdot P[|X_2| \le c] = 1 - \alpha \tag{2}$$

then:

(a) if $\max\{|X_1|, |X_2|\} \le c$, then accept $H_0$;

(b) if $\max\{|X_1|, |X_2|\} = M > c$, then let $i$ ($i = 1$ or 2) be such that $M = |X_i|$:

- if $X_i > 0$ conclude that $\mu_i > \mu_i^*$
- if $X_i < 0$ conclude that $\mu_i < \mu_i^*$,

then move to stage 2.

*Stage 2:* compare now the $\min\{|X_1|, |X_2|\} = m$ and the $(1 - \alpha/2)$-quantile of the Standard Normal (S.N.) distribution: $z_{1-\alpha/2}$. Let $j$ ($j = 1$ or 2) be such that $m = |X_j|$:

(a) if $m < z_{1-\alpha/2}$ then conclude that $\mu_j = \mu_j^*$

(b) if $m > z_{1-\alpha/2}$ then:

- if $X_j > 0$ conclude that $\mu_j > \mu_j^*$
- if $X_j < 0$ conclude that $\mu_j < \mu_j^*$.

**REMARKS**

1. The critical value $c$ is actually the $\left(\frac{1+\sqrt{1-\alpha}}{2}\right)$-quantile of the S.N. distribution. To see that, let $\beta$ denote the quantity

$$P[|X| \leq c] = P[-c \leq X \leq c]$$

where the rv $X$ has a S.N. distribution. In such case, $c$ is the $\left(\frac{1+\beta}{2}\right)$-quantile of the S.N. distribution. From equation (2) it follows that

$$\beta^2 = 1 - \alpha, \text{ that is } \beta = \sqrt{1-\alpha},$$

therefore $c$ is the $\left(\frac{1+\sqrt{1-\alpha}}{2}\right)$-quantile of the S.N. distribution.

2. The value of $c$ is calculated using the Bonferroni inequality, which means to assume that $X_1$ and $X_2$ are independent. If such rvs are not independent, the critical region is smaller, and therefore the probability of type I error is lower: for this reason Duncan test is a conservative one.

3. In the case of unknown marginal variances, they can be replaced by their estimations, under suitable assumptions regarding the sample size.

Insert Figure 1 here

### 2.2. IMPROVEMENT THROUGH THE ARCTANGENT DISTRIBUTION

Pollastri and Tornaghi (2004) stated and proved two results regarding the arctangent distribution, introduced in Zenga (1979). However, it is useful to recall the definition of such distribution.

**Definition 1** *A continuous rv $X$ is said to have the arctangent distribution depending on the parameter $a > 0$ if it has the following probability density function*

$$g(x;a) = \begin{cases} [\arctan(a)]^{-1} e^{-\frac{1}{2}x^2} \int_0^{ax} e^{-t^2/2} dt & x \geq 0 \\ 0 & \textit{otherwise.} \end{cases} \tag{3}$$

Now the two aforementioned results can be stated.

**Theorem 1** *Let $(X_1, X_2)$ be a bivariate rv with S.B.C.N. distribution depending on the correlation coefficient $\rho$ (see formula (1)). Then the distribution function of*

$M = \max\{|X_1|,|X_2|\}$ *is a mixture of two arctangent distributions with parameters* $a_1$ *and* $a_2$ *and weights equal to*

$$\pi_1 = \frac{\arctan(a_1)}{\pi/2} \qquad \textit{and} \qquad \pi_2 = \frac{\arctan(a_2)}{\pi/2},$$

*respectively, where the parameters* $a_1$ *and* $a_2$ *are:*

$$a_1 = \sqrt{\frac{1+\rho}{1-\rho}} \qquad \textit{and} \qquad a_2 = \sqrt{\frac{1-\rho}{1+\rho}}.$$

Hence, the probability density function of $M$ is:

$$f_M(t) = \begin{cases} g(t;a_1)\dfrac{\arctan(a_1)}{\pi/2} + g(t;a_2)\dfrac{\arctan(a_2)}{\pi/2} & t \geq 0 \\ 0 & otherwise \end{cases} \tag{4}$$

where $g(t;a_i)$ denotes the arctangent density function with parameter $a_i$, defined in (3).

**Theorem 2** *Let* $(X_1,X_2)$ *be a bivariate rv with S.B.C.N. distribution with correlation coefficient* $\rho$*. Then the probability density function of the rv* $V = \min\{|X_1|,|X_2|\}$ *is given by:*

$$f_V(x) = \begin{cases} 2(2\phi(x)) - f_M(x) & x \geq 0 \\ \textit{0} & \textit{otherwise} \end{cases} \tag{5}$$

*where* $\phi(x)$ *is the probability density function of the S.N. distribution and* $f_M$ *is the probability density function of* $M = \max\{|X_1|,|X_2|\}$.

Hence, $f_V(x)$ is a linear combination of the probability density function of a *Folded Standard Normal* rv and of the probability density function of the rv $M$.

### 2.3. THE DUNCAN-POLLASTRI TEST AND AN APPLICATION TO THE TRINOMIAL DISTRIBUTION

The results described in the previous section can be used to improve the Duncan test. Basically, the idea is that they provide the exact distributions of the test statistics:

$$M = \max\{|X_1|,|X_2|\} \qquad \text{and} \qquad V = \min\{|X_1|,|X_2|\}.$$

This allows to obtain “more exact” critical regions.

Let $(y_{1i}, y_{2i})$ with $(i = 1, 2, ..., n)$ be a simple random sample from a B.C.N. rv $\mathbf{Y} = (Y_1, Y_2)$ such that

$$(Y_1, Y_2) \sim N\left[(\mu_1, \mu_2); \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right].$$

Under the same assumptions described in Section 2 ($var(Y_1) = \sigma_1^2$ and $var(Y_2) = \sigma_2^2$ known), the Duncan-Pollastri test allows to verify the null hypothesis

$$H_0\colon (\mu_1 = \mu_1^*) \cap (\mu_2 = \mu_2^*)$$

against all the eight possible alternatives.

As for the Duncan test, the test statistic is based on the bivariate rv $(X_1, X_2)$, with components

$$X_1 = \frac{\bar{Y}_1 - \mu_1^*}{\sigma_1/\sqrt{n}} \quad \text{and} \quad X_2 = \frac{\bar{Y}_2 - \mu_2^*}{\sigma_2/\sqrt{n}},$$

which, under the null hypothesis $H_0$ has a S.B.C.N. distribution with correlation coefficient $\rho$. Such correlation coefficient can be estimated by the usual classical estimation techniques, for example maximum likelihood estimation.

Now, the procedure is divided into two stages.

*Stage 1:* for a fixed probability of type I error$\alpha$, using the estimate of $\rho$, the critical values $h(\alpha, |\rho|)$ and $k(\alpha, |\rho|)$ can be calculated through the tables provided in Pollastri and Riva (2012). Then:

(a) if $\max\{|X_1|, |X_2|\} \leq h(\alpha, |\rho|)$, then accept $H_0$;

(b) if $\max\{|X_1|, |X_2|\} = M > h(\alpha, |\rho|)$, then let $i$ ($i = 1$ or 2) be such that $M = |X_i|$:

- if $X_i > 0$ conclude that $\mu_i > \mu_i^*$
- if $X_i < 0$ conclude that $\mu_i < \mu_i^*$,

then move to Stage 2.

*Stage 2:* compare the $\min\{|X_1|, |X_2|\} = m$ and $k(\alpha, |\rho|)$.Let $j$ ($j = 1$ or 2) be such that $m = |X_j|$:

(a) if $m < k(\alpha, |\rho|)$ then conclude that $\mu_j = \mu_j^*$

(b) if $m > k(\alpha, |\rho|)$ then:

- if $X_j > 0$ conclude that $\mu_j > \mu_j^*$
- if $X_j < 0$ conclude that $\mu_j < \mu_j^*$.

As earlier claimed, the advantage of the Duncan-Pollastri test with respect to the Duncan test is that the former is based on the exact distributions of $M$ and $V$, while the latter is based on an approximation.

The Duncan-Pollastri test can be also applied for verifying statistical hypotheses on the parameters of a trinomial distribution.

Let $(X_1, X_2)$ be a rv with trinomial distribution depending on the parameters $n, p_1, p_2$. It is well-known that $X_1$ and $X_2$ have two binomial distributions with parameters $(n, p_1)$ and $(n, p_2)$, respectively. Then, for sufficiently large $n$, the two rvs

$$Z_1 = \frac{X_1 - np_1}{\sqrt{np_1(1-p_1)}} \quad \text{and} \quad Z_2 = \frac{X_2 - np_2}{\sqrt{np_2(1-p_2)}}$$

are the components of a bivariate rv $\mathbf{Z} = (Z_1, Z_2)$ with a S.B.C.N. distribution, that is

$$(Z_1, Z_2) \sim N\left[(0,0); \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right]$$

with

$$\rho = -\sqrt{\frac{p_1 p_2}{(1-p_1)(1-p_2)}}.$$

Then $M = \max\{|Z_1|, |Z_2|\}$ and $V = \min\{|Z_1|, |Z_2|\}$ can be considered. For Theorems 1 and 2 it follows that $M$ and $V$ have the probability density functions defined in (4) and (5), respectively. It is therefore possible to apply the aforementioned procedure and perform the Duncan-Pollastri test. It is worth highlighting that in this case, the Duncan-Pollastri test allows to verify the hypotheses regarding the parameters $p_1$ and $p_2$ with respect to two fixed values $p_1^*$ and $p_2^*$.

## 3. THE TWO-SAMPLE EXTENSION

In this section the Duncan-Pollastri test is extended to the case of two samples. Let us consider the following two situations.

1. Let a population be divided in three groups with proportions $p_1, p_2$, and $p_3$. Suppose that after some time, the proportions of the three groups are $p'_1, p'_2$, and $p'_3$. In some cases, it may be of interest to understand if (and possibly how) the proportions changed over time.

2. Assume that there are two different populations, each of them divided in three groups. Let $p_1, p_2$, and $p_3$ be the proportions in the first population and let $p'_1, p'_2$, and $p'_3$ be the corresponding proportions in the second population. The understanding if (and eventually how) the proportions in the two populations are different can find many applications for example in the market (or opinion) research field.

In both the situations, $p_3$ and $p'_3$ are not relevant. They can be obtained directly from the other proportions, since it is easy to see that:

$$p_3 = 1 - p_1 - p_2 \qquad \text{and} \qquad p'_3 = 1 - p'_1 - p'_2.$$

As mentioned before, it may be of interest to test the null hypothesis

$$H_0 : (p_1 = p'_1) \cap (p_2 = p'_2)$$

against all the eight possible alternatives:

$$\begin{array}{ll}
1) & (p_1 = p'_1) \cap (p_2 > p'_2) \\
2) & (p_1 = p'_1) \cap (p_2 < p'_2) \\
3) & (p_1 > p'_1) \cap (p_2 > p'_2) \\
4) & (p_1 > p'_1) \cap (p_2 < p'_2) \\
5) & (p_1 > p'_1) \cap (p_2 = p'_2) \\
6) & (p_1 < p'_1) \cap (p_2 = p'_2) \\
7) & (p_1 < p'_1) \cap (p_2 > p'_2) \\
8) & (p_1 < p'_1) \cap (p_2 < p'_2).
\end{array}$$

To do this, consider then two simple random samples from the two populations (or from the same population in different times) and the usual estimators of their proportions, that is

$$\hat{p}_1 = \frac{X_1}{n}, \ \hat{p}_2 = \frac{X_2}{n}, \ \text{ and } \ \hat{p}'_1 = \frac{X'_1}{n'}, \ \hat{p}'_2 = \frac{X'_2}{n'},$$

where:

- $X_i \sim Bin(n, p_i)$ is the number of elements of the $i$-th group in the first sample of size $n$ (with $i = 1, 2$);
- $X'_i \sim Bin(n', p'_i)$ is the number of elements of the $i$-th group in the second sample of size $n'$ (with $i = 1, 2$).

For sufficiently large $n$ and $n'$, the bivariate rv $(Z_1, Z_2)$ with components:

$$Z_1 = \frac{\hat{p}_1 - \hat{p}'_1}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n} + \frac{\hat{p}'_1(1-\hat{p}'_1)}{n'}}} \quad \text{and} \quad Z_2 = \frac{\hat{p}_2 - \hat{p}'_2}{\sqrt{\frac{\hat{p}_2(1-\hat{p}_2)}{n} + \frac{\hat{p}'_2(1-\hat{p}'_2)}{n'}}} \tag{6}$$

tends to a S.B.C.N with

$$\rho_{Z_1,Z_2} = \frac{-\frac{p_1 p_2}{n} - \frac{p'_1 p'_2}{n'}}{\sqrt{\frac{p_1(1-p_1)}{n} + \frac{p'_1(1-p'_1)}{n'}} \cdot \sqrt{\frac{p_2(1-p_2)}{n} + \frac{p'_2(1-p'_2)}{n'}}}. \tag{7}$$

Such correlation coefficient can be estimated by the following plug-in estimator:

$$\hat{\rho}_{Z_1,Z_2} = \frac{-\frac{\hat{p}_1 \hat{p}_2}{n} - \frac{\hat{p}'_1 \hat{p}'_2}{n'}}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n} + \frac{\hat{p}'_1(1-\hat{p}'_1)}{n'}} \cdot \sqrt{\frac{\hat{p}_2(1-\hat{p}_2)}{n} + \frac{\hat{p}'_2(1-\hat{p}'_2)}{n'}}}. \tag{8}$$

At this point, the Duncan-Pollastri procedure can be applied to decide in favour of one of the aforementioned hypotheses, using the test statistic based on $(Z_1, Z_2)$, since under the null hypothesis $H_0$, it holds true that:

$$(Z_1, Z_2) \sim N\left[(0,0); \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right].$$

For sake of completeness, as special case, if $n = n'$, the formula (7) becomes

$$\rho_{Z_1,Z_2} = \frac{-p_1 p_2 - p'_1 p'_2}{\sqrt{p_1(1-p_1) + p'_1(1-p'_1)} \cdot \sqrt{p_2(1-p_2) + p'_2(1-p'_2)}}$$

and therefore the estimate (8) becomes:

$$\hat{\rho}_{Z_1,Z_2} = \frac{-\hat{p}_1 \hat{p}_2 - \hat{p}'_1 \hat{p}'_2}{\sqrt{\hat{p}_1(1-\hat{p}_1) + \hat{p}'_1(1-\hat{p}'_1)} \cdot \sqrt{\hat{p}_2(1-\hat{p}_2) + \hat{p}'_2(1-\hat{p}'_2)}}$$

## 4. APPLICATIONS

This Section describes four applications of the Duncan-Pollastri test to real data. The first one regards the last presidential election in the United States of America, which took place in 2012. The remaining three analyses the level of trust of European citizens in official statistics of their country, regarding economic and social issues.

### 4.1. US ELECTION IN 2012

In November 2012 Americans voted for their President. Basically there were two candidates:

- Barack Obama (candidate for the Democratic Party);
- Mitt Romney (candidate for the Republican Party).

The following table shows the results of a survey, performed by Gallup (data are available at *http://www.gallup.com*). Both the sample sizes are 1063. The aim is to estimate the voting intention of Americans on two different dates: the former in August, the latter in October.

| | Obama | Romney | No opinion |
|---|---|---|---|
| Oct 27-28, 2012 | 54% | 34% | 12% |
| Aug 20-23, 2012 | 58% | 36% | 6% |

The Duncan-Pollastri test allows to decide whether the candidates' percentages have changed in the two surveys. Let $p_{Obama}$ and $p'_{Obama}$ be the percentages for Barack Obama in August and in October, respectively; and let $p_{Romney}$ and $p'_{Romney}$ be the corresponding percentages for Romney. The null hypothesis $H_0$ claims that the percentages in the two surveys have not changed, that is:

$$H_0: \ (p_{Obama} = p'_{Obama}) \cap (p_{Romney} = p'_{Romney}).$$

From the data in the above table, the estimates of the bivariate rv $(Z_1, Z_2)$, defined in (6) can be easily calculated:

$$z_1 = \frac{0.58 - 0.54}{\sqrt{\dfrac{0.58 \cdot 0.42}{1063} + \dfrac{0.54 \cdot 0.46}{1063}}} = 1.18593,$$

$$z_2 = \frac{0.36 - 0.34}{\sqrt{\frac{0.36 \cdot 0.64}{1063} + \frac{0.34 \cdot 0.66}{1063}}} = 0.9669.$$

The estimate of the correlation coefficient is:

$$\hat{\rho}_{Z_1,Z_2} = \frac{-\frac{0.58 \cdot 0.36}{1063} - \frac{0.54 \cdot 0.34}{1063}}{\sqrt{\frac{0.58 \cdot 0.42}{1063} + \frac{0.54 \cdot 0.46}{1063}} \cdot \sqrt{\frac{0.36 \cdot 0.64}{1063} + \frac{0.34 \cdot 0.66}{1063}}} = -0.83.$$

Now, using the tables provided in Pollastri and Riva (2012) it holds that:

*Stage 1:* $\max\{|z_1|, |z_2|\} = M = 1.8593$

(a) if $\alpha < 0.10$ then $\max\{|z_1|, |z_2|\} = 1.8593 \leq h(\alpha, |-0.83|)$, therefore the test accepts $H_0$ and it stops;

(b) if $\alpha \geq 0.10$, as $M = z_1 = 1.8593 > 0$, the test concludes that $p_{Obama} > p'_{Obama}$ and the stage 2 follows.

*Stage 2:* $\min\{|z_1|, |z_2|\} = m = 0.9669.$
if $\alpha \geq 0.10$, as $m = 0.9669 < k(\alpha, |-0.83|)$, the test concludes that $p_{Romney} = p'_{Romney}$.

The test therefore provides the following conclusion, depending on the value of $\alpha$:

- if $\alpha < 0.10$, accept the null hypothesis:

$$H_0 : \ (p_{Obama} = p'_{Obama}) \cap (p_{Romney} = p'_{Romney});$$

- if $\alpha \geq 0.10$, accept the alternative hypothesis:

$$(p_{Obama} > p'_{Obama}) \cap (p_{Romney} = p'_{Romney}).$$

### 4.2. TRUST ON THE NATIONAL STATISTICS IN EUROPEAN COUNTRIES

The remaining three examples are applications to data collected by *TNS Opinion & Social* in *"Special Eurobarometer 323 - Europeans' knowledge of economic indicators"*. In this report, some European citizens have been asked to evaluate the level of trust in official statistics released by national statistical offices. The exact question was:

> ”How much trust do you have in official statistics, for example unemployment, inflation or economic growth?”

The results for different countries have been compared in the following examples.

**EXAMPLE 1**

The first example compares the results for Bulgaria (BG) and Italy (IT). The data of the survey are summarised in the following table:

| Country | $n$ | Tend to trust ($\hat{p}_1$) | Tend NOT to trust ($\hat{p}_2$) |
|---|---|---|---|
| BG | 1025 | 47% | 47% |
| IT | 1039 | 42% | 41% |

The Duncan-Pollastri test can be applied. The required estimates are:

$$z_1 = 2.288, \ \ z_2 = 2.750, \ \ \hat{\rho} = -0.799.$$

In this case it is worth noting that if the value of $\alpha$ is set to 0.01:

- the Duncan test accepts the null hypothesis:
  $$(p_1^{BG} = p_1^{IT}) \cap (p_2^{BG} = p_2^{IT}),$$
  since the critical value $c = 2.81 > 2.750 = \max\{|z_1|, |z_2|\}$;
- while the Duncan-Pollastri test accepts the alternative:
  $$(p_1^{BG} > p_1^{IT}) \cap (p_2^{BG} > p_2^{IT}),$$
  since
  $$\max\{|z_1|, |z_2|\} = 2.750 \quad > \quad 2.7479 = h(0.01, |-0.799|),$$
  and
  $$\min\{|z_1|, |z_2|\} = 2.288 \quad < \quad 2.2334 = k(0.01, |-0.799|).$$

This example shows clearly that in some cases the two tests can decide to accept different hypotheses.

**EXAMPLE 2**

The second example compares the results for Italy (IT) and Poland (PL). As before, the data of the survey are summarised in the a table:

| Country | $n$ | Tend to trust ($\hat{p}_1$) | Tend NOT to trust ($\hat{p}_2$) |
|---|---|---|---|
| IT | 1039 | 42% | 41% |
| PL | 1000 | 45% | 41% |

Applying the Duncan-Pollastri test, the required estimates are:

$$z_1 = -1.367, \ z_2 = 0, \ \hat{\rho} = -0.732.$$

For $\alpha = 0.05$, the test leads to accept the null hypothesis:

$$(p_1^{IT} = p_1^{PL}) \cap (p_2^{IT} = p_2^{PL}),$$

since

$$\max\{|z_1|, |z_2|\} = 1.367 \quad < \quad 2.1730 = h(0.05, |-0.732|).$$

**EXAMPLE 3**

The last example regards Italy (IT) and France (FR). The data of the survey are:

| Country | $n$ | Tend to trust ($\hat{p}_1$) | Tend NOT to trust ($\hat{p}_2$) |
|---|---|---|---|
| IT | 1039 | 42% | 41% |
| FR | 1027 | 40% | 56% |

As in the other cases, applying the Duncan-Pollastri test, the required quantities are estimated:

$$z_1 = 0.924, \ z_2 = -6.899, \ \hat{\rho} = -0.816.$$

For $\alpha = 0.05$, the test leads to accept the alternative hypothesis:

$$(p_1^{IT} = p_1^{FR}) \cap (p_2^{IT} < p_2^{FR}),$$

since

$$\max\{|z_1|, |z_2|\} = 6.899 \quad > \quad 2.1425 = h(0.05, |-0.816|),$$

and

$$\min\{|z_1|, |z_2|\} = 0.924 \quad < \quad 1.6657 = k(0.05, |-0.816|).$$

## 5. CONCLUSIONS

This paper describes a two-stage test to compare the probabilities of two trinomial distributions. The Duncan-Pollastri test is an improvement of the Duncan test, based on the exact distributions of the two test statistics. This improvement allows to obtain a "more exact" region of rejection for a fixed value of probability of type I error. Such test is not difficult to perform: the required estimation procedures can be easily implemented and do not require large resources. The Duncan-Pollastri test can be very useful in proximity of political elections or to verify the effectiveness of an advertising campaign.

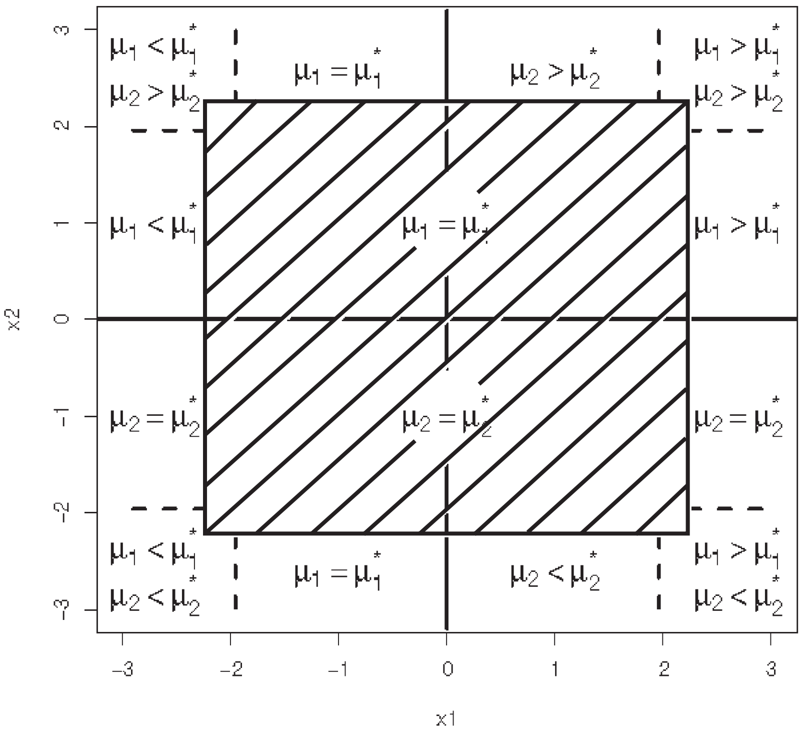


**Figure 1: The acceptance region for the Duncan Test with $\alpha = 0.05$, being $c = 2.236$ and $z_{1-\alpha/2} = 1,96$**

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