GENERALISED LINEAR MODELS WITH VARIABLES SUBJECT TO POST RANDOMISATION METHOD

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Abstract. *The Post Randomisation Method (PRAM) is a disclosure avoidance method, where values of potentially identifying categorical variables are perturbed via some known probability mechanism. The main goal is to reduce the risk of identification, but the use of misclassified data raises issues about validity of statistical inference. In this paper, we develop and implement an expectation-maximisation (EM) algorithm to obtain unbiased parameter estimates of generalised linear models (GLMs) when data are subject to PRAM, and thus account for the effects of PRAM and preserve data utility. We derive new standard errors of the estimates, and evaluate the effects of the level of perturbation and sample size on the reduction of bias through a number of simulation studies and by applying the proposed methodology to a dataset from the 1993 Current Population Survey.*

Keywords: *Statistical disclosure control, Generalised linear models, EM algorithm, PRAM.*

1. INTRODUCTION

Statistical agencies face the challenge of providing high-quality data products to aid valid statistical research and policy-making, while also guarding against the risk of disclosing confidential information about individual respondents. Statistical Disclosure Control (SDC) methods aim at finding the best compromise between maximising data utility and minimising disclosure risk. For more details on SDC methodology and its importance to official statistics, see Fienberg and Slavković (2010); Hundepool et al. (2012); Ramanayake and Zayatz (2010); Wil-

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lenborg and de Waal (1996).

Microdata are sets of records containing detailed information on individual respondents and there is a substantial demand for these high quality data products nowadays. There are many SDC methods for microdata, and when applied they typically lead to publishing of one or more altered datasets by introducing bias and variance to the original data. The Post Randomisation Method (PRAM) is a SDC method originally proposed by Gouweleeuw et al. (1998). The main idea behind PRAM is to publish redacted data after the values of categorical variables in the original dataset have been misclassified by a known probability mechanism.

PRAM has, however, seen a limited use in practice due to a number of unresolved issues related to data utility such as biased inference. Since PRAM can be regarded as a type of misclassification, the PRAMed data suffer from similar bias issues as described in the measurement error literature. For example, according to Frost and Thompson (2000), the regression dilution bias or attenuation is present when a single predictor has a measurement error but that error is unbiased and independent of the response variable and of the underlying value. When a single categorical predictor is misclassified, Buonaccorsi et al. (2005) point out that there will be bias in the naive estimates of the regression coefficient for other non-misclassified covariates unless the misclassified variable is independent of those. Furthermore, in the multivariate setting and when the response is also measured with error, there does not seem to be general characterisation of direction and magnitude of bias; for more on measurement error models and related discussions, see Buonaccorsi (2010). Since the parameter estimates and summary statistics based on PRAMed data display some such biases, they need to be adjusted to take the effects of PRAM into account in order to obtain valid inference.

To address the bias in PRAMed data, for example, Gouweleeuw et al. (1998) proposed an unbiased moment estimator for frequency counts, van den Hout and van der Heijden (2002) proposed estimates of odds ratios for data subject to PRAM, and most recently Woo and Slavković (2012) proposed an expectation-maximisation (EM) algorithm to obtained unbiased estimates of binary logistic regression coefficients. In the related literature on randomised response, the issue of unbiased estimation was addressed by van den Hout and Kooiman (2006) who proposed an EM algorithm to estimate the linear regression model with covariates subject to randomised response, and van den Hout et al. (2007) proposed using a Newton-Raphson method to estimate the logistic regression model with response variable subject to randomised response.

In this paper, we develop and implement an EM algorithm to obtain asymptotically unbiased estimators, that is the maximum likelihood estimators of parameters in generalised linear models (GLMs), when chosen categorical variables are subject to PRAM. We build on the basic ideas from the "EM by method of weights" developed by Ibrahim (1990) for GLMs with covariates missing at random, and on the approach proposed by van den Hout and Kooiman (2006) for linear regression with covariates subject to randomised response. Our EM algorithm obtains unbiased estimates of GLMs when either one or more categorical covariates or the categorical response variable is subject to PRAM and when both the categorical covariates and the categorical response variable are subject to PRAM. The latter is a more difficult problem, and has received little attention in statistics literature. In this paper, we assume that the covariates are independent. Our results show that even if this assumption may not seem reasonable, the use of the proposed algorithm significantly improves the data utility in comparison to relying on the unadjusted inference. This work extends the methodology and results presented in Woo and Slavković (2012) on binary logistic regression, adds new simulation results, and adds a new risk-utility tradeoff analysis for a real data application.

The rest of this paper is organised as follows. In Section 2 we introduce PRAM and present the EM-type methodology to obtain asymptotically unbiased estimates of GLMs when variables are subject to PRAM. Three different GLM cases are considered: (1) when categorical covariates are subject to PRAM, (2) when categorical response variable is subject to PRAM, and (3) when both the categorical covariates and the categorical response variable are subject to PRAM. Through a number of simulations studies in Section 3, we evaluate the performance of the proposed algorithms using examples from logistic and Poisson regressions and we evaluate the effects of the level of perturbation and sample size on the reduction of bias. In Section 4 we apply the proposed methodology to data from the 1993 Current Population Survey (CPS) (data from The National Bureau of Economic Research (2013)), and provide additional discussion in Section 5.

2. GENERALISED LINEAR MODELS WITH VARIABLES SUBJECT TO PRAM

2.1 OVERVIEW OF PRAM

We start with a brief description of PRAM. Let W denote a categorical variable in the original file to which PRAM will be applied, and let W^* denote the same categorical variable in the perturbed file. The levels of W and W^* are $\{w_1, \cdots, w_J\}$. The probability that an original value $W = w_j$ is changed into the value $W^* = w_k$ is denoted by the transition (or misclassification) probability $p_{\mathrm{W}_{jk}}$ = $P(W^* = w_k \mid W = w_j)$ for all $j,k = 1,...,J$. This probability mechanism is described by $\boldsymbol{P}_W$, a $J \times J$ transition matrix, i.e., PRAM matrix, with entries $p_{\mathrm{W}_{jk}}$ such that $\sum_{k=1}^{J} p_{\mathrm{W}_{jk}} = 1$. If we let $\boldsymbol{T}_W$ be the $J \times 1$ vector of frequencies of the J levels of W observed in the original file, and $\boldsymbol{T}_{W^*}$ be the $J \times 1$ vector of frequencies in the perturbed file, then by the Law of Total Probability, frequency estimation with PRAM can be presented as

$$\hat{\boldsymbol{T}}_W = \left(\boldsymbol{P}_W^\top\right)^{-1} \boldsymbol{T}_{W^*}. \tag{1}$$

As already pointed out, this is a very specific type of misclassification and as such can be related to more general studies of measurement error models (e.g., see Buonaccorsi (2010)). Equation (1) shows that we require knowledge about the values of the transition probabilities in the PRAM matrix to be able to estimate the frequency counts in the original file. Indeed, if an analyst wishes to perform any kind of statistical analysis on the perturbed file, some sort of adjustment based on the values in the PRAM matrix needs to be made in order to obtain results similar to the results from analyses performed on the original file. Next, we investigate this issue as it pertains to estimating coefficients in generalised linear models.

2.2 GENERALISED LINEAR MODEL ESTIMATION WITH VARIABLES SUBJECT TO PRAM

Let Y denote a response variable that comes from an exponential family distribution,

$$f(y;\theta,\psi) = \exp\{\frac{y\theta - b(\theta)}{a(\psi)} + c(y,\psi)\}$$

for some functions $a(.), b(.), c(.)$, where θ is the canonical parameter, and ψ is the dispersion parameter. Let $\mathbf{X}$ denote a design matrix with p covariates, and

$\boldsymbol{\beta}$ denote a $(p+1) \times 1$ vector of the regression parameters. A generalised linear model (GLM) can be expressed as,

$$E(y(i)|\boldsymbol{x(i)}) = \mu(i) = g^{-1}(\boldsymbol{x(i)\beta}), \tag{2}$$

where the mean, $\mu(i)$, of $y(i)$ for observation i depends on the linear predictor $\eta(i) = \boldsymbol{x(i)\beta}$, i.e., $g(\mu(i)) = \eta(i) = \boldsymbol{x(i)\beta}$, where $g(.)$ is the link function.

Since under the application of PRAM only the perturbed data are published, estimating GLMs with variables subject to PRAM can be viewed as an incomplete data problem, with the perturbed, released data associated with the true, unreleased data. The EM algorithm is a standard iterative procedure for computing maximum likelihood estimates with incomplete data (Dempster et al., 1977; Wu, 1983). In each iteration of the algorithm, the E step derives the conditional expectation of the complete data likelihood given the observed data and the current estimates of the parameters, and the M step follows by maximising the conditional expectation of the likelihood with respect to the parameters. We develop and implement an EM algorithm to obtain unbiased estimates of $\boldsymbol{\beta}$ in GLMs for three cases: (1) categorical covariates subject to PRAM; (2) categorical response variable subject to PRAM; and (3) both the categorical covariates and the response variable subject to PRAM. Below we present three versions of the proposed algorithm to highlight the differences across these three cases.

2.2.1. CATEGORICAL COVARIATE SUBJECT TO PRAM

First we describe an EM algorithm to obtain unbiased estimates of regression coefficients in GLMs when categorical covariates are subject to PRAM. This method is similar to the “EM by method of weights" proposed by Ibrahim (1990), which is used to estimate parameters in GLMs with missing covariates; a related algorithm for the linear regression model with covariates subject to randomised response was proposed by van den Hout and Kooiman (2006). For ease of exposition, we present the case where PRAM is applied to one categorical covariate, but the methodology generalises to applying PRAM independently to multiple categorical covariates; for an example see Section 3.3.2.

Let $\mathbf{X} = (W, \mathbf{Z})$, where $\mathbf{X}$ denotes all the covariates and W denotes the categorical covariate to which PRAM is applied. Let W^* denote the observed and released version of private W with levels $\{w_1, ..., w_J\}$, and $\mathbf{Z}$ denotes the covariates which are not subject to PRAM and can be both categorical and continuous. Let $\boldsymbol{P}_\mathrm{W}$ be the J x J PRAM matrix that contains the misclassification probabilities

$p_{\mathrm{W}_{jk}} = P(W^* = w_k | W = w_j)$, and $\pi_j^* = P(W^* = w_j)$ and $\pi_j = P(W = w_j)$ be the marginal probabilities; the subscripts indicate the level of the categorical variable.

The joint distribution of $(\mathbf{x}(\mathbf{i}), y(i))$ is specified via the conditional distribution of $y(i)$ given $\mathbf{x}(\mathbf{i})$ and the distribution of $\mathbf{x}(\mathbf{i})$ which is a joint distribution of $(w(i), \mathbf{z}(\mathbf{i}))$, for observations $i = 1, ..., n$. The complete data log-likelihood can be expressed as

$$\begin{aligned} \ell(\phi; W, \mathbf{Z}, y) &= \sum_{i=1}^{n} \ell(\phi; \mathbf{x}(\mathbf{i}), y(i)) \\ &= \sum_{i=1}^{n} \left\{ \ell_{y(i)|\mathbf{x}(\mathbf{i})}(\boldsymbol{\beta}) + \ell_{w(i)|\mathbf{z}(\mathbf{i})}(\boldsymbol{\pi}) + \ell_{\mathbf{z}(\mathbf{i})}(\boldsymbol{\gamma}) \right\}, \end{aligned} \quad (3)$$

where $\boldsymbol{\phi} = (\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\pi})$, and the distribution of W is multinomial with parameter $\boldsymbol{\pi}$.

At iteration ν of the EM algorithm (see *EM Algorithm I*), where $\phi^{(\nu)}$ is the value of parameter ϕ at iteration ν, the E-step can be written as

$$\begin{aligned} Q\left(\phi^{(\nu)} | \phi^{(\nu-1)}\right) &= \sum_{i=1}^{n} E\left(\ell(\phi; \mathbf{x}(\mathbf{i}), y(i)) \,|\text{data}, \phi^{(\nu-1)}\right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{J} q_j^{(\nu)}(i) \ell\left(\phi^{(\nu-1)}; \mathbf{x}(\mathbf{i}), y(i)\right). \end{aligned} \quad (4)$$

The first part of (3) is the log-likelihood of the GLM, and for example, when we have a single covariate the last two parts reduce to the log-likelihood of a multinomial distribution with parameter $\boldsymbol{\pi}$. The M-step maximises (4) which can be done via a weighted regression, by creating a “new" dataset, with each observation i taking on all possible values of W, i.e. $(W(i) = w_1), (W(i) = w_2), ..., (W(i) = w_J)$, with weights

$$q_j^{(\nu)}(i) = P(W(i) = w_j | y(i), w^*(i), \mathbf{z}(\mathbf{i}), \phi^{(\nu-1)}), \quad (5)$$

where w^* is the observed value of the variable subject to PRAM. Using Bayes' rule, the weights $q_j^{(\nu)}$ are

$$P\left(W = w_j | W^* = w_k, Y, \mathbf{Z}, \phi^{(\nu-1)}\right) = \frac{P\left(Y | w_j, \mathbf{Z}, \phi^{(\nu-1)}\right) p_{\mathrm{W}_{jk}} \pi_j^{(\nu-1)}}{\sum_{l=1}^{J} P\left(Y | w_l, \mathbf{Z}, \phi^{(\nu-1)}\right) p_{\mathrm{W}_{lk}} \pi_l^{(\nu-1)}} . \quad (6)$$

Note that $P(Y|.)$ in (6) is the probability density for the GLM. In the algorithm by Ibrahim (1990) for GLMs with missing covariates, the inner sum in (4) is taken over all levels of W, but only for observations with missing covariates. In our setting, since PRAM is applied to all observations, the inner sum is taken for all observations. In the algorithm by van den Hout and Kooiman (2006), the weights in (4) are multiplied by the number of levels J but with no justification.

EM Algorithm I: Initial values $\boldsymbol{\beta}^{(0)}$ can be the estimates of $\boldsymbol{\beta}$ from the regression of $Y \sim \mathbf{X}^*$, where $\mathbf{X}^* = (W^*, \mathbf{Z})$. $\boldsymbol{\pi}^*$ can be used as the initial estimate $\boldsymbol{\pi}^{(0)}$. Thus $\boldsymbol{\phi}^{(0)} = (\boldsymbol{\beta}^{(0)}, \boldsymbol{\pi}^{(0)})$.
E-step:
Compute $q_j^{(v)}(i)$ using (6) for $i = 1, ..., n$ and $j = 1, ..., J$.
M-step:
Carry out weighted regression with weights $q_j^{(v)}(i)$, using standard software.
Update $\boldsymbol{\phi}^{(v)}$*:*
$\boldsymbol{\beta}^{(v)} = \hat{\boldsymbol{\beta}}$ from weighted regression.
$\pi_j^{(v)} = \sum_{i=1}^{n} q_j^{(v)}(i)/n$ for $j = 1, \cdots, J$.
With the updated $\phi^{(v)}$, new weights can be computed in the E-step, and the algorithm continues until convergence.

Note that for the initial estimate of $\boldsymbol{\pi}$ in the *EM Algorithm I* above, one can also use $\hat{\boldsymbol{\pi}} = (\boldsymbol{P}_{\mathrm{W}}^{\top})^{-1}\boldsymbol{\pi}^*$ which eliminates an update step for $\boldsymbol{\pi}$; see *EM Algorithm III*. Furthermore, the proposed weights assume that the covariates are independent; see Appendix A. The initial estimate of γ from equation (3) can come from the marginal distribution of Z, and is not updated since it is a parameter that describes the covariates that are not subject to PRAM.

2.2.2 RESPONSE VARIABLE SUBJECT TO PRAM

Here we present the second case when a categorical response variable is subject to PRAM. We focus on a binary response variable and binary logistic regression, but the key ideas extend to response variables with more than two levels and to multinomial logistic regression. Let Y denote the binary response variable to which PRAM is applied, with Y^* denoting the observed and released version of private Y. Let $\boldsymbol{P}_{\mathrm{Y}}$ be the PRAM matrix that contains the probabilities, with $p_{\mathrm{Y}_{jk}} = P(Y^* = k|Y = j)$, $k, j \in \{0, 1\}$. Following the method proposed in Section 2.2.1, the parameter $\boldsymbol{\pi}$ in the complete data log-likelihood (3) can be estimated directly since **X** is not subject to PRAM. Thus, at iteration ν of the algorithm (see *EM Algorithm II*), the E-step simplifies to

$$Q\left(\phi^{(\nu)}|\phi^{(\nu-1)}\right) = \sum_{i=1}^{n}\sum_{j=0}^{1} P(Y(i)=j|y^*(i),\mathbf{x}(\mathbf{i}),\beta^{(\nu-1)})\left\{\ell_{y_i|\mathbf{x_i}}\left(\beta^{(\nu-1)}\right)\right\} \quad (7)$$

The M-step maximises (7) via a weighted regression, by creating a “new" dataset, with each observation i taking on $(Y=0)$, $(Y=1)$ with weights $r_j^{(\nu)}(i) = P(Y(i) = j|y^*(i),\mathbf{x}(\mathbf{i}),\beta^{(\nu-1)})$. Using Bayes' rule, the weights $r_j^{(\nu)}(i)$ can be computed as

$$P\left(Y=j|Y^*=k,\mathbf{X},\beta^{(\nu-1)}\right) = \frac{P\left(Y=j|\mathbf{X},\beta^{(\nu-1)}\right)p_{\mathrm{Y}_{jk}}}{\sum_{l=0}^{1} p_{\mathrm{Y}_{lk}} P\left(Y=l|\mathbf{X},\beta^{(\nu-1)}\right)} \quad . \qquad (8)$$

EM Algorithm II runs as follows:

EM Algorithm II: Initial values $\beta^{(0)}$ can be the estimates of β from the regression of $Y^* \sim \mathbf{X}$, where Y^* is the response variable subject to PRAM.
E-step:
Compute $r_j^{(\nu)}(i)$ using (8) for $i=1,\ldots,n$ and $j=0,1$.
M-step:
Carry out weighted regression with weights $r_j^{(\nu)}(i)$, using standard software.
Update $\beta^{(\nu)}$:
$\beta^{(\nu)} = \hat{\beta}$ from weighted regression.
With the updated $\beta^{(\nu)}$, new weights can be computed in the E-step, and the algorithm continues until convergence.

2.2.3 COVARIATE AND RESPONSE SUBJECT TO PRAM

As the amount of data accumulate in the public domain, and as agencies seek to share more data, there is a likely need to mask a greater number of variables. In this section we describe how to obtain unbiased parameter estimates of GLMs when both the categorical covariates and the categorical response variable are subject to PRAM. This case is more complex than the other two and very important in practice but has received little attention in the literature. The key difference from the previous two algorithms is that the weights in (9) below are more complex than the weights presented in (6) and (8); see Appendix C for more details.

The weighted regression is done by creating a “new" dataset with each observation i taking on all possible values of Y and W, i.e. $(Y=0),(Y=1)$ and $(W=w_1),(W=w_2),...,(W=w_J)$ with weights $s_{ml}^{(\nu)}(i) = P(Y(i)=m, W(i) = w_l | Y^*(i)=k, W^*(i)=w_j, \mathbf{Z}, \phi^{(\nu-1)})$. The weights can be computed as

$$s_{ml}^{(\nu)} = \frac{p_{\mathrm{Y}_{mk}} P\left(Y=m|W=w_l,\mathbf{Z},\phi^{(\nu-1)}\right)}{\sum_{a=0}^{1} p_{\mathrm{Y}_{ak}} P\left(Y=a|W=w_l,\mathbf{Z},\phi^{(\nu-1)}\right)} \times \frac{p_{\mathrm{W}_{lj}} \pi_l \sum_{b=0}^{1} p_{\mathrm{Y}_{bk}} P\left(Y=b|W=w_l,\mathbf{Z},\phi^{(\nu-1)}\right)}{\sum_{c=1}^{J} p_{\mathrm{W}_{cj}} \pi_c \sum_{d=0}^{1} p_{\mathrm{Y}_{dk}} P\left(Y=d|W=w_c,\mathbf{Z},\phi^{(\nu-1)}\right)} . \tag{9}$$

Note that we assume that the W and Z are independent but not with the response. When more than one covariate is misclassified via PRAM, we assume independent misclassification mechanisms. The proposed algorithm runs as follows:

EM Algorithm III: Initial values $\boldsymbol{\beta}^{(0)}$ can be the estimates of $\boldsymbol{\beta}$ from the regression of of $Y^* \sim \mathbf{X}^*$, where $\mathbf{X}^* = (W^*, \mathbf{Z})$. $\boldsymbol{\pi}$ can be estimated by $\boldsymbol{\pi}^{(0)} = (\boldsymbol{P}_{\mathrm{W}}^{\top})^{-1}\boldsymbol{\pi}^*$. Thus $\phi^{(0)} = (\boldsymbol{\beta}^{(0)}, \boldsymbol{\pi}^{(0)})$.
E-step:
Compute $s_{ml}^{(\nu)}(i)$ using (9) for $i=1,...,n$, $m=0,1$ and $l=1,\cdots,J$.
M-step:
Carry out weighted regression with weights $s_{ml}^{(\nu)}(i)$, using standard software.
Update $\boldsymbol{\beta}^{(\nu)}$:
$\boldsymbol{\beta}^{(\nu)} = \hat{\boldsymbol{\beta}}$ from weighted regression
$\boldsymbol{\pi}_l^{(\nu)} = \sum_{i=1}^{n}\sum_{m=0}^{1} s_{ml}^{(\nu)}(i)/n$ for $l=1,\cdots,J$.
With the updated $\boldsymbol{\beta}^{(\nu)}$, a new dataset with new weights can be computed in the E-step, and the algorithm continues until convergence.

The marginal distribution of W can be estimated by the unbiased estimator $\hat{\boldsymbol{\pi}} = (\boldsymbol{P}_{\mathrm{W}}^{\top})^{-1}\boldsymbol{\pi}^*$ instead of using the ML estimate approach as seen in Algorithm I, but either one is acceptable.

3. NUMERICAL ILLUSTRATIONS OF THE PROPOSED ALGORITHMS

We carried out a number of simulation studies to demonstrate the effect of PRAM on the MLEs of regression coefficients in GLMs for the above three cases by reporting the inference on the original data, the unadjusted inference and our adjusted inference on PRAMed data. We evaluate the performance of the proposed algorithms and the effects of the level of perturbation and sample size on the reduction of bias using examples from logistic and Poisson regressions. The main goal is to enable the users to obtain the same inference with PRAMed data as they would have if they had access to the original confidential data, and more specifically to reduce the bias that arises due to application of PRAM. We focus our evaluation on the relative bias of the estimates with respect to the true β. We also report the standard errors, mean squared errors (MSEs), and coverage probabilities. In general, our algorithms perform extremely well in terms of low relative bias; they offer solid guarantees that the users would reach the same inference with PRAMed data as they would have if they had access to the original data given suitable sample sizes. Our methodology clearly outperforms the unadjusted inference. The simulations were done in R (R Core Team, 2013). For each example, we ran 500 simulations, with varying sample size $n = 100, 1000, 10000$.

3.1 EXAMPLE I: LOGISTIC REGRESSION

In the first example, we evaluate the proposed EM algorithms in estimating the logistic regression model when variables are subject to PRAM. As before, we let Y be the binary response variable, $\mathbf{X}$ a design matrix with p covariates and n observations, and $\mathbf{x}(\mathbf{i}) = (x_0(i), \cdots, x_p(i))$ a vector of the covariates for observation $i = 1, \cdots, n$. The logistic regression model can be written as

$$E\left(y(i)|\mathbf{x}(\mathbf{i})\right) = \frac{\exp\left(\mathbf{x}(\mathbf{i})\boldsymbol{\beta}\right)}{1+\exp\left(\mathbf{x}(\mathbf{i})\boldsymbol{\beta}\right)} \tag{10}$$

where $\boldsymbol{\beta} = (\beta_0, \beta_1, ..., \beta_p)^\top$ are the regression coefficients and parameters of interest. We fix $x_{0,i} = 1$ for $i = 1, \cdots, n$ so β_0 is the intercept.

For each simulation, we generate a random sample from a population of interest, with $\boldsymbol{\beta} = (0.5, 0.5)^\top$, a binary covariate X with $x_1(i)$ sampled from Bernoulli(0.4), and $y(i)$ sampled from Bernoulli$\left(\frac{\exp(\mathbf{x}(\mathbf{i})\boldsymbol{\beta})}{1+\exp(\mathbf{x}(\mathbf{i})\boldsymbol{\beta})}\right)$. We treat this as the confidential original data, and fit a logistic regression to obtain the estimate of the slope, $\hat{\beta}_1$. PRAM is then applied to each of the following three cases:

1. PRAM applied to $x_1(i)$ to obtain $x_1^*(i)$.
2. PRAM applied to $y(i)$ to obtain $y^*(i)$.
3. PRAM applied to both $x_1(i)$ and $y(i)$ to obtain $x_1^*(i)$ and $y^*(i)$, respectively.

In all three cases, the released data contain original variables that are not subject to PRAM and information on the variables to which PRAM was applied; i.e., the original values of the variables subject to PRAM are considered private and unobserved. We use PRAM matrices of the following form

$$\boldsymbol{P} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix},$$

where the level of perturbation is varied by varying the value of $p = 0.8, 0.9$. A lower value for p indicates a higher level of perturbation, and lower values than these are rarely considered in practice.

After the application of PRAM, we perform both the unadjusted and adjusted inference. We fit logistic regression to the PRAMed data and compute the estimate of the slope, $\hat{\beta}_{1,\text{noadjust}}$ and evaluate its performance in terms of relative bias ($\frac{\hat{\beta}_{1,\text{noadjust}}-\beta_1}{\beta_1}$), standard error, MSE, and coverage probability over 500 simulations; see Table 1 in the row labeled "$\hat{\beta}_{1,\text{noadjust}}$". Next, we fit logistic regression to the same PRAMed data but using the appropriate proposed EM algorithm to obtain the estimate of the slope, i.e., $\hat{\beta}_{1,\text{adjust}}$. We report the relative bias, standard error, MSE, and coverage probability of $\hat{\beta}_{1,\text{adjust}}$ over 500 simulations in Table 1 in the row labeled "$\hat{\beta}_{1,\text{adjust}}$". The following stopping criterion $|\beta_{1,adjust}^{(v+1)} - \beta_{1,adjust}^{(v)}| < 10^{-4}$ was used to ensure convergence of the EM algorithms.

To estimate the covariance matrix, we use the inverse of the observed information matrix. For Algorithm I, following the method described in Ibrahim et al. (2005), the estimated observed information matrix is given by

$$\begin{aligned} I\left(\hat{\phi}\right) = & - \ddot{Q}\left(\hat{\phi}|\hat{\phi}^{(v)}\right) - \sum_{i=1}^{n}\sum_{j=1}^{J} q_j(i)\, S(i)\left(\hat{\phi}|\mathbf{x}(\mathbf{i}), y(i)\right) S(i)\left(\hat{\phi}|\mathbf{x}(\mathbf{i}), y(i)\right)^{\top} \\ & + \sum_{i=1}^{n} \dot{Q}(i)\left(\hat{\phi}|\hat{\phi}^{(v)}\right) \dot{Q}(i)\left(\hat{\phi}|\hat{\phi}^{(v)}\right)^{\top} \end{aligned} \tag{11}$$

where $\ddot{Q}(\phi|\phi^{(v)}) = \sum_{i=1}^{n}\sum_{j=1}^{J} q_j(i) \frac{\partial^2 \ell(\phi|\mathbf{x}(\mathbf{i}), y(i))}{\partial\phi\partial\phi^{\top}}$, $\dot{Q}(i)(\phi|\phi^{(v)}) = \sum_{j=1}^{J} q_j(i) \frac{\partial \ell(\phi|\mathbf{x}(\mathbf{i}), y(i))}{\partial\phi}$, and $S(i)(\phi|\mathbf{x}(\mathbf{i}), y(i)) = \frac{\partial \ell(\phi|\mathbf{x}(\mathbf{i}), y(i))}{\partial\phi}$. To estimate the covariance matrices when

Table 1: Relative bias, standard error, mean squared error, & coverage probability of MLEs for logistic regression with variables subject to PRAM. Case 1: PRAM applied to covariate. Case 2: PRAM applied to response variable. Case 3: PRAM applied to covariate and response variable

			p=0.9				p=0.8			
			Rel Bias	**SE**	**MSE**	**Cov Prob**	**Rel Bias**	**SE**	**MSE**	**Cov Prob**
Case 1	$n = 100$	$\hat{\beta}_{1,\text{noadjust}}$	-0.1858	0.4600	0.2202	0.936	-0.3684	0.4563	0.2421	0.936
		$\hat{\beta}_{1,\text{adjust}}$	0.0758	0.5359	0.2886	0.944	0.3467	1.3710	1.9999	0.982
	$n = 1000$	$\hat{\beta}_{1,\text{noadjust}}$	-0.1905	0.1351	0.0273	0.898	-0.4040	0.1426	0.0611	0.706
		$\hat{\beta}_{1,\text{adjust}}$	0.0387	0.1595	0.0258	0.946	0.0452	0.1807	0.0332	0.950
	$n = 10000$	$\hat{\beta}_{1,\text{noadjust}}$	-0.2156	0.0428	0.0134	0.284	-0.4280	0.0436	0.0477	0.002
		$\hat{\beta}_{1,\text{adjust}}$	0.00049	0.0502	0.0025	0.954	-0.0055	0.0563	0.0032	0.946
Case 2	$n = 100$	$\hat{\beta}_{1,\text{noadjust}}$	-0.1572	0.4401	0.1999	0.934	-0.3945	0.4123	0.2089	0.922
		$\hat{\beta}_{1,\text{adjust}}$	0.1600	0.6344	0.4089	0.952	0.1977	0.9435	0.9000	0.946
	$n = 1000$	$\hat{\beta}_{1,\text{noadjust}}$	-0.2406	0.1319	0.0319	0.846	-0.4600	0.1342	0.0709	0.626
		$\hat{\beta}_{1,\text{adjust}}$	0.0042	0.1831	0.0335	0.954	-0.0041	0.2517	0.0634	0.938
	$n = 10000$	$\hat{\beta}_{1,\text{noadjust}}$	-0.2462	0.0447	0.0171	0.218	-0.4580	0.0421	0.0542	0.000
		$\hat{\beta}_{1,\text{adjust}}$	-0.0054	0.0576	0.0033	0.956	-0.0081	0.0788	0.0062	0.956
Case 3	$n = 100$	$\hat{\beta}_{1,\text{noadjust}}$	-0.4455	0.4581	0.2595	0.934	-0.6577	0.4353	0.2976	0.880
		$\hat{\beta}_{1,\text{adjust}}$	0.2889	1.6957	2.8961	0.986	0.3256	2.4878	6.6133	0.906
	$n = 1000$	$\hat{\beta}_{1,\text{noadjust}}$	-0.4055	0.1333	0.0589	0.648	-0.7013	0.1258	0.1388	0.218
		$\hat{\beta}_{1,\text{adjust}}$	0.0076	0.2311	0.0534	0.962	-0.0476	0.4113	0.1698	0.948
	$n = 10000$	$\hat{\beta}_{1,\text{noadjust}}$	-0.4053	0.0410	0.0428	0.002	-0.6808	0.0401	0.1175	0
		$\hat{\beta}_{1,\text{adjust}}$	0.0011	0.0706	0.0050	0.940	0.0050	0.1299	0.0169	0.952

running EM algorithms II and III, expression in (11) is adjusted, with weights $q_j(i)$ replaced by weights from (8) and (9) for cases two and three, respectively.

From the summary of the results in Table 1, we observe that across all three cases the estimates $\hat{\beta}_{1,\text{noadjust}}$ have much higher relative bias than $\hat{\beta}_{1,\text{adjust}}$, and within each case the relative bias of $\hat{\beta}_{1,\text{noadjust}}$ minimally increases with the increase in the sample size. As expected, the relative bias of $\hat{\beta}_{1,\text{noadjust}}$ increases with a higher level of perturbation, e.g., for Case 1 and $n = 10000$, the relative bias is -0.2156 and -0.4280 when $p = 0.90$ and $p = 0.80$ respectively. Case 3, where both the response variable and the covariate are misclassified, produces estimates which have significantly higher relative bias than the first two cases. We also note the relative bias shows attenuation, that is the $\hat{\beta}_{1,\text{noadjust}}$ is closer to 0 than the true β_1. For Case 1 this is in line with results from the measurement error literature on attenuation phenomenon when covariates are misclassified. We notice a similar behavior in Cases 2 and 3, when the response and both the response and the covariate are subject to PRAM.

In contrast, the $\hat{\beta}_{1,\text{adjust}}$ estimates, using the proposed EM algorithms, perform well in reducing the relative bias and thus clearly outperform $\hat{\beta}_{1,\text{noadjust}}$. For example, in Case 1 with $n = 10000$ and $p = 0.9$, the relative bias of $\hat{\beta}_{1,\text{adjust}}$ is 0.0005 compared to the relative bias of $\hat{\beta}_{1,\text{noadjust}}$, which is -0.2156. Furthermore, as the sample size increases, there is a significant reduction in relative bias due to the use of the proposed algorithms, e.g., in Case 1, the relative bias of $\hat{\beta}_{1,\text{adjust}}$ is 0.0758, 0.0387, and 0.0005 compared to the relative bias of $\hat{\beta}_{1,\text{noadjust}}$ with values of -0.1858, -0.1905, and -0.2156 for $n = 100$, $n = 1000$, and $n = 10000$, respectively. Figures 1, 2, and 3 also illustrate improved performance of $\hat{\beta}_{1,\text{adjust}}$ across different sample sizes for Cases 1, 2, and 3, respectively. As expected, the algorithms perform slightly better when $p = 0.9$ compared to when $p = 0.8$, in terms of producing the smaller relative bias since the first perturbation introduces less misclassification.

When we first considered the issue of GLM parameter estimation with variables subject to PRAM, our main objective in maintaining data utility was to obtain unbiased estimates of the regression coefficients. While our algorithms produce estimators with no or very small biases, these estimators may have slightly higher standard errors than in the cases when no adjustments are made. Thus other measures such as the mean-squared error (MSE) that capture both the bias and variance may be also useful in judging the quality of the estimation. From Table 1, we can see that the MSEs for the adjusted inference is also much smaller than for the unadjusted inference unless we are dealing with a small sample size. For example, for $n = 100$, while the estimates $\hat{\beta}_{1,\text{adjust}}$ have smaller relative biases, their MSEs are typically higher in all three cases. As sample size increases, $\hat{\beta}_{1,\text{adjust}}$ have smaller MSEs than $\hat{\beta}_{1,\text{noadjust}}$. This is mostly due to the significant reduction in bias as sample size increases. We also note that as the sample size increases, the coverage probabilities of the estimates $\hat{\beta}_{1,\text{adjust}}$ are close to the nominal 95% levels, while the coverage probabilities of $\hat{\beta}_{1,\text{noadjust}}$ fall far away from the nominal 95% levels.

Figures 1, 2, and 3 display the ML estimates of $\hat{\beta}_{1,\text{adjust}}$ for the first 50 simulated samples for sample sizes $n = 100, n = 1000$, and $n = 10000$, and $p = 0.9$ using EM algorithms I, II, and III, respectively. We note that in Figure 3 for $n = 100$, when both the covariate and the response variables are subject to PRAM, there is one abnormal simulated sample where the ML estimate is around 15 and is far away from the true value of $\beta_1 = 0.5$. Furthermore, out of 500 simulated samples, 5 had ML estimates of $\hat{\beta}_{1,\text{adjust}}$ much larger than β_1 (i.e., greater than 5). These atypical samples contribute to the large standard errors and MSEs for all three cases when $n = 100$. For the official statistics data products this is not a likely sample size as it is too small, thus such extreme behavior may not be observed. However, it is important to investigate the algorithms' performance in extreme cases as well. This phenomenon requires a more careful study, but as an initial practical solution when the proportion of such extreme cases is small we suggest that those runs could be dropped from the analysis.

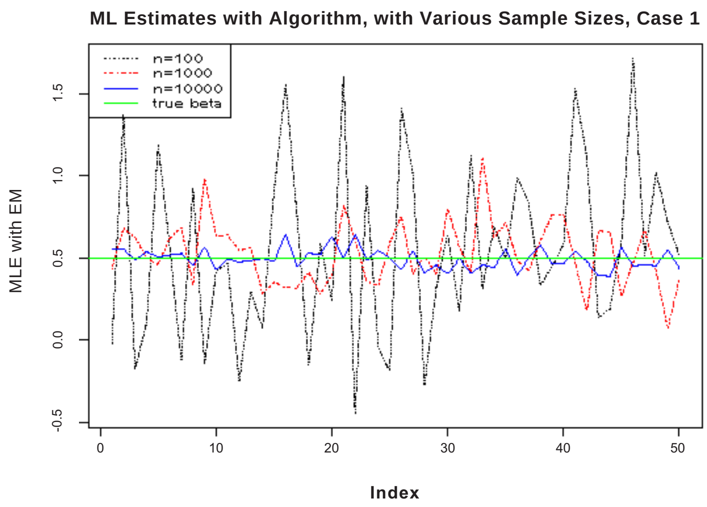

Figure 1: Case 1: Plot of ML estimates $\hat{\beta}_{1,adjust}$ in logistic regression for the first 50 simulated samples using EM algorithm I, with various sample sizes and $p = 0.9$

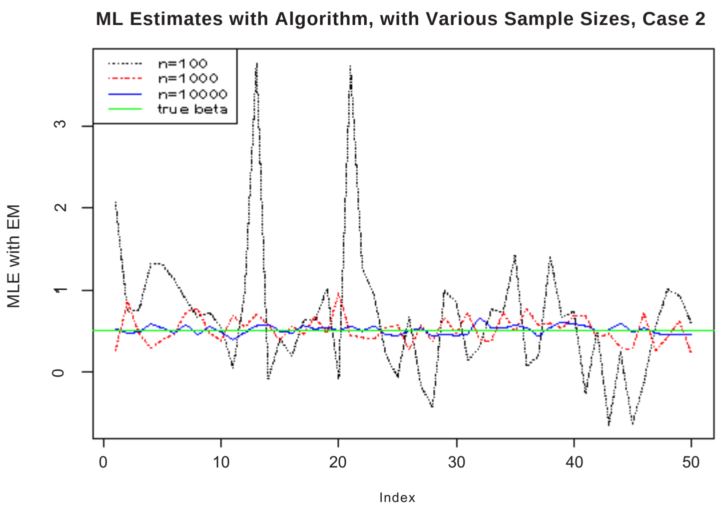

Figure 2: Case 2: Plot of ML estimates $\hat{\beta}_{1,adjust}$ in logistic regression for the first 50 simulated samples using EM algorithm II, with various sample sizes and $p = 0.9$

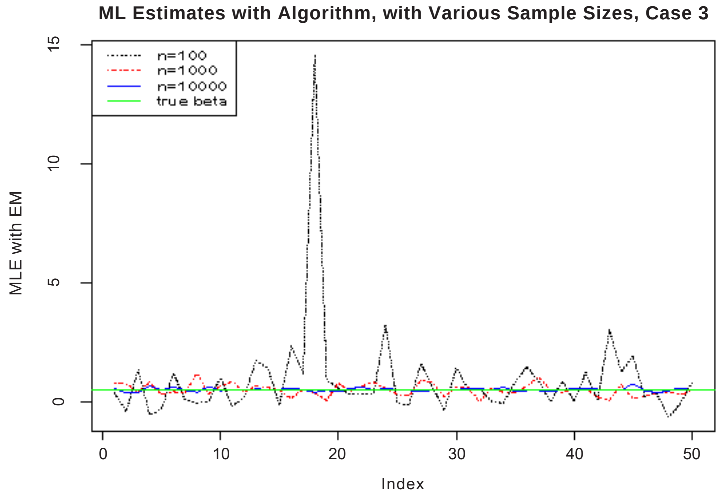

Figure 3: Case 3: Plot of ML estimates $\hat{\boldsymbol{\beta}}_{1,\text{adjust}}$ in logistic regression for the first 50 using EM algorithm III, with various sample sizes and p = 0.9

3.2 EXAMPLE II: POISSON REGRESSION

In the second example, we evaluate the performance of EM algorithm I in estimating the Poisson regression model when categorical covariates are subject to PRAM. The Poisson regression model can be written as

$$E\left(y(i)|\mathbf{x(i)}\right) = \exp(\mathbf{x(i)}\boldsymbol{\beta}), \tag{12}$$

where $\boldsymbol{\beta} = (\beta_0, \beta_1, ..., \beta_p)^\top$ are the regression coefficients and parameters of interest. We fix $x_0(i) = 1$ so β_0 is the intercept. In order to apply PRAM to responses those must be categorical too, and in the Poisson regression this could be done by transforming the count response data into a multinomial variable and fitting a multinomial logistic regression. Note that in the Poisson regression example reported here, we apply PRAM only to categorical covariates.

Similarly to the logistic regression experiments, for each simulation, we generate a random sample from a population of interest with $\boldsymbol{\beta} = (0.2, 0.6)^\top$, a binary covariate X with $x_1(i)$ sampled from Bernoulli(0.5), and $y(i)$ sampled from Poisson$(\mathbf{x(i)}\boldsymbol{\beta})$. We treat this as the confidential original data, and fit Poisson regression to obtain the estimate $\hat{\beta}_1$. PRAM is applied to the covariate $x_1(i)$ to obtain $x_1^*(i)$. We evaluate the effect of PRAM on Poisson regression by computing and reporting the relative bias, standard error, MSE, and coverage probability of $\hat{\beta}_{1,\text{noadjust}}$ with respect to β_1 over 500 simulations, and of $\hat{\beta}_{1,\text{adjust}}$ after applying the proposed EM algorithm to adjust for the affect of PRAM.

The results on Poisson regression in Table 2 show similar trends to what was discussed in the case of logistic regression. $\hat{\beta}_{1,\text{adjust}}$ outperforms $\hat{\beta}_{1,\text{noadjust}}$ in terms of reducing the relative bias and having coverage probability being close to the nominal 95% level, although the standard error does increase slightly. In general, the MSE is also smaller but unlike the results in the case of logistic regression, the mean squared error of $\hat{\beta}_{1,\text{adjust}}$ is comparable to the mean squared error of $\hat{\beta}_{1,\text{noadjust}}$ even for the small sample sizes, e.g., $n = 100$.

Table 2: Relative bias, standard error, mean squared error, & coverage probability of MLEs for Poisson regression with covariate subject to PRAM

		p=0.9				p=0.8			
		Rel Bias	**SE**	**MSE**	**Cov Prob**	**Rel Bias**	**SE**	**MSE**	**Cov Prob**
$n = 100$	$\hat{\beta}_{1,\text{noadjust}}$	-0.1993	0.1664	0.0420	0.878	-0.4076	0.1642	0.0868	0.670
	$\hat{\beta}_{1,\text{adjust}}$	-0.0015	0.2079	0.0432	0.942	-0.0365	0.2565	0.0663	0.952
$n = 1000$	$\hat{\beta}_{1,\text{noadjust}}$	-0.2054	0.0507	0.0178	0.322	-0.4123	0.0504	0.0637	0.004
	$\hat{\beta}_{1,\text{adjust}}$	0.0004	0.0616	0.0038	0.948	-0.0060	0.0747	0.0056	0.952
$n = 10000$	$\hat{\beta}_{1,\text{noadjust}}$	-0.2068	0.0165	0.0157	0	-0.4113	0.0157	0.0612	0
	$\hat{\beta}_{1,\text{adjust}}$	0.0006	0.0206	0.0004	0.946	0.0006	0.0239	0.0006	0.954

3.3 EFFECTIVENESS OF EM ALGORITHM I

In this section, we consider the effectiveness of the proposed EM Algorithm I in more detail by varying the probabilities of success and distribution of covariates, and by applying PRAM to multiple covariates.

3.3.1 VARYING THE PROBABILITY OF SUCCESS & DISTRIBUTION OF COVARIATES

Here, we evaluate the performance of EM Algorithm I subject to varying the probabilities of success for a binary response variable and varying the distribution of a binary covariate. For $n = 1000$, one binary covariate $x_1(i)$ is sampled from Bernoulli(π) and $y(i)$ is sampled from Bernoulli($\frac{\exp(\mathbf{x}(\mathbf{i})\boldsymbol{\beta})}{1+\exp(\mathbf{x}(\mathbf{i})\boldsymbol{\beta})}$), where $\boldsymbol{\beta} = (1, \beta_1)^\top$ with following values of $\beta_1 = \{-2, -0.5, 0.5, 2\}$ and $\pi = \{0.1, 0.2, 0.3, 0.4, 0.5\}$. PRAM is applied to $x_1(i)$ to obtain $x_1^*(i)$ using the same form of the PRAM matrix as in the previous sections and varying the perturbation proportions as $p = 0.9$ and $p = 0.8$. We compute the estimates of the regression coefficients $\hat{\beta}_{1,\text{adjust}}$ over 500 simulations of model (10), and report the relative bias, standard error, MSE, and coverage probability of $\hat{\beta}_{1,\text{adjust}}$ in Table 3.

Table 3: Relative bias, standard error, mean squared error, & coverage probability of MLEs in logistic regression, when accounting for PRAM applied to a single categorical covariate, with distribution of covariate and probability of success varied

		p=0.9				p=0.8			
		Rel Bias	**SE**	**MSE**	**Cov Prob**	**Rel Bias**	**SE**	**MSE**	**Cov Prob**
$\beta_1 = -2$	$\pi = 0.1$	0.0244	0.5091	0.2598	0.944	-0.0682	0.5395	0.2957	0.948
	$\pi = 0.2$	0.0127	0.2668	0.0713	0.958	-0.0009	0.4143	0.1716	0.950
	$\pi = 0.3$	0.0094	0.2185	0.0478	0.940	0.0011	0.3113	0.0969	0.958
	$\pi = 0.4$	0.0136	0.1973	0.0391	0.952	0.0218	0.2687	0.0727	0.946
	$\pi = 0.5$	0.0059	0.1875	0.0352	0.946	-0.0031	0.2583	0.0667	0.948
$\beta_1 = -0.5$	$\pi = 0.1$	-0.0519	0.3850	0.1509	0.952	-0.1257	0.5439	0.3116	0.948
	$\pi = 0.2$	0.0139	0.2331	0.0545	0.950	0.0008	0.3186	0.1015	0.964
	$\pi = 0.3$	-0.0051	0.1866	0.0348	0.942	0.0075	0.2615	0.0684	0.958
	$\pi = 0.4$	0.0077	0.1863	0.0348	0.944	0.0450	0.2253	0.0528	0.948
	$\pi = 0.5$	0.0186	0.1738	0.0306	0.948	0.0199	0.2291	0.0529	0.950
$\beta_1 = 0.5$	$\pi = 0.1$	0.0970	0.5382	0.29903	0.958	0.0647	0.7395	0.5510	0.960
	$\pi = 0.2$	0.0379	0.2956	0.0888	0.954	0.1908	0.4653	0.2456	0.940
	$\pi = 0.3$	0.0327	0.2311	0.0545	0.952	0.0284	0.3161	0.1007	0.950
	$\pi = 0.4$	0.0123	0.2136	0.0458	0.944	0.0168	0.2771	0.0771	0.950
	$\pi = 0.5$	0.00729	0.1919	0.0369	0.962	-0.0010	0.2601	0.0677	0.942
$\beta_1 = 2$	$\pi = 0.1$	0.2161	1.6830	2.8791	0.922	-0.2154	0.9703	0.9878	0.950
	$\pi = 0.2$	0.0975	0.9131	0.8433	0.934	-0.0319	0.9448	0.8937	0.964
	$\pi = 0.3$	0.0645	0.5867	0.3484	0.954	0.0658	0.7616	0.5843	0.954
	$\pi = 0.4$	0.0489	0.4604	0.2164	0.958	0.0428	0.6734	0.4553	0.958
	$\pi = 0.5$	0.0100	0.3719	0.1384	0.956	0.0647	0.5804	0.3410	0.950

The results suggest that the proposed algorithm works better when the distribution of the response variable and covariate are less skewed. For all values of β_1, as π approaches 0.5, the relative bias, standard error, and MSE decrease. Apart from the case when $\beta_1 = 2$ and $\pi = 0.1$, the relative biases are all low. In all cases, the coverage probability is at the nominal 95% level.

3.3.2 APPLYING PRAM TO TWO INDEPENDENT COVARIATES

We also evaluate the effectiveness of EM Algorithm I when PRAM is applied independently to more than one covariate. We consider two binary covariates with $x_1(i)$ and $x_2(i)$ sampled independently from Bernoulli(0.45) and Bernoulli(0.55), and $y(i)$ sampled from Bernoulli$(\frac{\exp(\mathbf{x(i)}\boldsymbol{\beta})}{1+\exp(\mathbf{x(i)}\boldsymbol{\beta})})$ with $\boldsymbol{\beta} = (-0.6, 0.8, -0.3)^\top$. We ran 500 simulations for various sample sizes $n = 100, 1000$ and $n = 10000$, and perturbation levels of $p = 0.8, 0.9$ as before. The results for a logistic regression model comparing the unadjusted and adjusted inference are displayed in Table 4.

The estimates of $\hat{\beta}_{\text{k,adjust}}$ for $k = 1, 2$, as with the case of one covariate, are consistently less biased than when no adjustment is applied. For example, when $n = 1000$ and $p = 0.8$, the relative bias for $\hat{\beta}_{1,\text{adjust}}$ is 0.0061 while for $\hat{\beta}_{1,\text{noadjust}}$ the relative bias is -0.4081, and for $\hat{\beta}_{2,\text{adjust}}$ the relative bias is -0.0698 while for

$\hat{\beta}_{2,\text{noadjust}}$ the relative bias is -0.4615. As before, the improvement in relative bias is more pronounced as the sample size increases, e.g., smaller relative bias with increasing sample size. When no adjustment is made, the coverage probability of $\hat{\beta}_{\text{k,noadjust}}$ decreases, further supporting the case that an adjustment is needed if a user is to make a valid inference with PRAMed data. The case of dependent covariates is considered next.

Table 4: Relative bias, standard error, mean squared error, & coverage probability of MLEs for logistic regression with two independent covariates

		p=0.9				p=0.8			
		Rel Bias	**SE**	**MSE**	**Cov Prob**	**Rel Bias**	**SE**	**MSE**	**Cov Prob**
$n = 100$	$\hat{\beta}_{1,\text{noadjust}}$	-0.2173	0.4354	0.2368	0.922	-0.3731	0.4419	0.3345	0.884
	$\hat{\beta}_{1,\text{adjust}}$	0.0097	0.5753	0.3311	0.942	0.1497	08457	0.7376	0.946
	$\hat{\beta}_{2,\text{noadjust}}$	-0.1879	0.4327	0.2226	0.966	-0.4509	0.4250	0.3840	0.948
	$\hat{\beta}_{2,\text{adjust}}$	0.0656	0.5699	0.3291	0.954	0.0374	0.8234	0.6794	0.946
$n = 1000$	$\hat{\beta}_{1,\text{noadjust}}$	-0.2213	0.1354	0.0673	0.698	-0.4081	0.1269	0.1826	0.262
	$\hat{\beta}_{1,\text{adjust}}$	-0.0168	0.1724	0.0300	0.950	0.0061	0.2194	0.0482	0.940
	$\hat{\beta}_{2,\text{noadjust}}$	-0.2230	0.1244	0.0652	0.934	-0.4615	0.1225	0.2280	0.826
	$\hat{\beta}_{2,\text{adjust}}$	-0.0103	0.1587	0.0253	0.958	-0.0698	0.2122	0.0499	0.954
$n = 10000$	$\hat{\beta}_{1,\text{noadjust}}$	-0.2078	0.0426	0.0450	0.022	-0.4076	0.0416	0.1679	0
	$\hat{\beta}_{1,\text{adjust}}$	-0.0014	0.0541	0.0029	0.962	0.0024	0.0714	0.0051	0.940
	$\hat{\beta}_{2,\text{noadjust}}$	-0.2154	0.0397	0.0480	0.664	-0.4188	0.0430	0.1772	0.140
	$\hat{\beta}_{2,\text{adjust}}$	-0.0018	0.0506	0.0026	0.950	-0.0006	0.0740	0.0055	0.942

4. APPLICATION TO 1993 CPS DATASET

We implemented the proposed methodology on data from the 1993 Current Population Survey (CPS); see The National Bureau of Economic Research (2013) for the data. The dataset contains 48842 records on 8 categorical variables. We performed logistic regression for *salary* ($0 = <$\$50,000 or $1 = >$\$50,000) on the covariates *Sex* ($0 =$ Female or $1 =$ Male), *Race* ($0 =$ Non White or $1 =$ White), and *Marital Status* ($0 =$ Married or $1 =$ Unmarried). The parameter estimates $\hat{\beta}_k$, for $k = 0,1,2,3$ from fitting the logistic regression with the original data are displayed in the first line of Table 5, labeled as "O.D.".

We present results on the following four cases: 1) *marital status* subject to PRAM; 2) *salary* subject to PRAM; 3) *marital status* and *salary* subject to PRAM; and 4) *race* and *marital status* subject to PRAM. In each case, we considered two perturbation levels $p = 0.9$ and $p = 0.8$. We ran 500 simulations for each case with fitting the standard logistic regression (without adjustment) on the PRAMed data. The mean estimates of $\hat{\beta}_{\text{k,noadjust}}$ for $k = 0,1,2,3$ are computed and are reported in Table 5 in the rows labeled "$\hat{\beta}_{\text{k,noadjust}}$". Then using the proposed algorithms I, II and III, we compute and report the mean MLEs in Table 5 in the rows labeled "$\hat{\beta}_{\text{k,adjust}}$". The algorithms typically converged at around 15 steps, based on the criterion $|\beta_k^{(\nu+1)} - \beta_k^{(\nu)}| < 10^{-4}$.

Since we do not know the true value of the regression parameters, the relative biases reported in Table 5 are now computed with respect to $\hat{\beta}$ from the original data as $(\frac{\hat{\beta}_{\text{k,noadjust}}-\hat{\beta}_k}{\hat{\beta}_k})$ and $(\frac{\hat{\beta}_{\text{k,adjust}}-\hat{\beta}_k}{\hat{\beta}_k})$ for $\hat{\beta}_{\text{k,noadjust}}$ and $\hat{\beta}_{\text{k,adjust}}$, respectively. Similarly, we define the mean bias as $\frac{\hat{\beta}_{\text{k,noadjust}}-\hat{\beta}_k}{500}$ and $\frac{\hat{\beta}_{\text{k,adjust}}-\hat{\beta}_k}{500}$, where 500 is the number of simulations. In terms of these measures, Case 2, when only the response variable *salary* was perturbed, appears to provide the best results followed by Case 1. The proposed algorithms are less effective for cases 3 and 4, and this is not surprising given that PRAM is applied to more than one variable. For example for cases 1 through 4, the mean bias of $\hat{\beta}_{3,adjust}$ (*martial status*) is 0.0604, -0.0116, 0.1303, and 0.1981 and significantly lower compared to $\hat{\beta}_{3,noadjust}$ with values 0.7157, 0.8627, 1.2585, and 0.7104, respectively. The same phenomenon can be observed for the relative bias, e.g., -0.0261, 0.0050, -0.0563, and -0.0855 for $\hat{\beta}_{3,adjust}$ and -0.3089, -0.3724, -0.5433, -0.3067 for $\hat{\beta}_{3,noadjust}$. These reductions in both types of bias when using the proposed algorithms are very promising.

Table 5: Parameter Estimates from Original Data (O.D.), data subject to PRAM without EM algorithms ($\hat{\beta}_{\text{k,noadjust}}$), and data subject to PRAM with EM Algorithms ($\hat{\beta}_{\text{1,adjust}}$). Average ML estimates with standard errors in brackets for O.D. Average ML estimates with relative bias, with respect to $\hat{\beta}_k$, in parentheses for $\hat{\beta}_{\text{k,noadjust}}$ and $\hat{\beta}_{\text{k,noadjust}}$. Case 1: *marital* subject to PRAM; Case 2: *salary* subject to PRAM; Case 3: both *marital* and *salary* subject to PRAM; Case 4: *race* and *marital* subject to PRAM

		$\hat{\beta}_0$	$\hat{\beta}_1$ **(gender)**	$\hat{\beta}_2$ **(race)**	$\hat{\beta}_3$ **(marital status)**
	O.D.	-0.8585 [0.0453]	0.2855 [0.0325]	0.3925 [0.0384]	-2.3166 [0.0309]
Case 1	$\hat{\beta}_{\text{k,noadjust}}$	-1.4458 (0.6841)	0.7398 (1.5912)	0.4400 (0.1210)	-1.6009 (-0.3089)
	$\hat{\beta}_{\text{k,adjust}}$	-1.0477 (0.2204)	0.3851 (0.3489)	0.4249 (0.0825)	-2.2562 (-0.0261)
Case 2	$\hat{\beta}_{\text{k,noadjust}}$	-0.5475 (0.3623)	0.1550 (-0.4571)	0.2323 (-0.4081)	-1.4539 (-0.3724)
	$\hat{\beta}_{\text{k,adjust}}$	-0.7785 (0.0932)	0.2138 (-0.2511)	0.3745 (-0.0459)	-2.3282 (0.0050)
Case 3	$\hat{\beta}_{\text{k,noadjust}}$	-1.2807 (0.4918)	0.7283 (1.5510)	0.2721 (-0.3068)	-1.0581 (-0.5433)
	$\hat{\beta}_{\text{k,adjust}}$	-1.2469 (0.4524)	0.4372 (0.5313)	0.444 (0.1312)	-2.1863 (-0.0563)
Case 4	$\hat{\beta}_{\text{k,noadjust}}$	-1.3202 (0.5378)	0.7545 (1.6427)	0.2302 (-0.4135)	-1.6062 (-0.3067)
	$\hat{\beta}_{\text{k,adjust}}$	-1.2534 (0.4560)	0.5026 (0.7604)	0.4253 (0.0836)	-2.1185 (-0.0855)

An additional reasonable explanation for decrease in performance in cases 3 and 4, is that algorithms I and III assume independence of covariates, and in this dataset the covariates are associated; i.e., the chi-squared test of independence for all two-way tables for the covariates *Sex*, *Race*, and *Marital Status*, imply strong association with all p-values close to 0. Algorithm II does not require the distribution of the covariate to be specified. Furthermore, these data are unbalanced and have low proportions; for example, 33% of the sample were female, and only 14% of the sample were non-white. Based on the simulation results from Section 3.3., we can expect that the algorithms' effectiveness will drop although they still lead to much better results than when no adjustment is considered at all.

4.1 DISCLOSURE RISK ASSESSMENT

When applying any SDC methodology, both the disclosure risk and the data utility should be evaluated. The main focus of our paper is on preserving data utility of microdata subject to PRAM when fitting a GLM. We now briefly discuss disclosure risk assessment and how a PRAM matrix may be chosen in this setting, but much is left to further investigation. Ideally, the chosen PRAM matrix is one that maximises data utility under some predetermined levels of disclosure risk set by the statistical agency; see Woo (2013) for more details.

Traditionally, a record is considered safe whenever a certain combination of scores on identifying variables occur at least d times, that is as suggested by de Wolf and van Gelder (2004), a safe record can be linked with at least d records in the population. If done randomly, the probability that the record is linked correctly in the population is less than or equal to d^{-1}, that is the risk of disclosure is at most d^{-1} (e.g., see equation (15)). A traditional measure of disclosure risk for releasing PRAMed variables, also proposed by de Wolf and van Gelder (2004), calculates the conditional probability that given a record with level k in the perturbed file, the original level was k as well. This conditional probability is $P(Y = k|Y^* = k)$, where Y represents the variable subject to PRAM and can be estimated by

$$\hat{R}_{\text{PRAM},k} = \frac{p_{Y_{kk}} T_{Y,k}}{\sum_l p_{Y_{lk}} T_{Y,l}},$$

where $T_{Y,l}$ are the frequency counts in the sample for variable Y, level l. However, given our methodology, the weights obtained upon the convergence of the EM algorithm can be also used to estimate this risk.

For example, for the CPS data for Case 2, where the response variable *Salary* was subject to PRAM and regressed against *Sex*, *Race*, and *Marital Status*, we want to evaluate the following disclosure risk:

$$R_{\text{PRAM},k,m} = P(Y = k, X = m | Y^* = k, X = m). \tag{13}$$

Here, X is a three-dimensional vector for the covariates *Sex*, *Race*, and *Marital Status*, and m is a three-dimensional vector of their observed values. Using Bayes theorem, (13) can be estimated by

$$\hat{R}_{\text{PRAM},k,m} = \frac{p_{Y_{kk}} P(Y = k | X = m)}{\sum_l p_{Y_{lk}} P(Y = l | X = m)}. \tag{14}$$

Given our proposed methodology, however, the risk in (13) can be be evaluated using $P(Y = k | Y^* = k, X = m, \beta^{(\nu-1)})$, which is estimated by using the weights from (8) obtained upon convergence of EM Algorithm II; the other cases are similar. Note that this potentially new measure of disclosure risk uses more information (i.e., estimated β) than the estimate in (14), and as such we anticipate that the risk measure based on the weights should be "better" than (14). This warrants a further careful study that is beyond the scope of this paper.

In our running example for the CPS data Case 2, a record is safe whenever

$$\hat{R}_{\text{PRAM},k} \leq \frac{T_{Y,k,m}}{d} \tag{15}$$

where $T_{Y,k,m}$ is the frequency count for the four-way table of $Y = k$ and $X = m$. For example, if the agency decides that the threshold is $d = 100$, the only set of records that are potentially at risk are records that have $Salary =$ more than $\$50,000$, $Sex =$ "$Female$", $Race =$ "$Non-White$", and $Marital =$ "$Unmarried$", since there are only 82 such records in the dataset and the frequencies for the other combinations of these variables are greater than 100. Using (14) to estimate $R_{\text{PRAM},k}$, the risk is estimated to be 0.2236, which is less than $\frac{82}{100}$, thus satisfying (15). Using the weights from (8), the risk is estimated to be 0.2099 on average across the 500 simulations, and also satisfies the set threshold in (15). The two risks are similar in this case leading to the same inference but further careful investigation of advantages and disadvantages of the two ways to measure the disclosure risk is needed. Both risk measures assume independence of the covariates, but it is possible that by providing new algorithms to capture possible associations between the covariates or by changing the levels of perturbation we may see more significant difference across the two measures.

How can we potentially capture the risk-utility tradeoff here? Note that as the diagonal entries of a PRAM matrix p_{kk} decrease, a record is more likely to be safe according to the constraint (15). Our results from Section 3 indicate that we obtain unbiased estimates but the MSE of the estimators tend to increase as p_{kk} decreases. Since for larger sample sizes the increase in MSE due to the use of the proposed algorithms is negligible, the choice of entries in a PRAM matrix should be determined by the disclosure risk constraint (15). However, for the smaller sample size, we may need to consider more carefully a data utility question such as: What is an acceptable MSE?

5. DISCUSSION

PRAM misclassifies the original confidential data to reduce the risk of disclosure of sensitive attributes and thus potentially makes the microdata more readily available to the users outside the official statistical agencies. However, the released data may lead to biased and unreliable inference. In this paper, we propose an EM algorithm to obtain unbiased parameter estimates of regression coefficients in GLMs with data subject to PRAM, and thus account for the effects of PRAM and preserve data utility. We use the quality of inference as the yardstick by comparing the estimates of regression coefficients in GLMs computed on the data subject to PRAM, with and without using the proposed EM algorithms. As expected, from our examples with logistic and Poisson regressions, the estimates are biased when performing the unadjusted inference on PRAMed data, thus affecting data utility, and in some instances may lead to wrong inference. However, by using the proposed EM methodology, we can obtain asymptotically unbiased estimates, nominal or close to nominal coverage probabilities, smaller MSEs, and thus more valid statistical inference.

Based on the simulation results, the proposed algorithms produce accurate regression coefficient estimates especially with larger sample sizes (when $n = 1000$ or $n = 10000$) given that the underlying distribution is not severely skewed. This is in part due to the fact that the weights computed in the E-step are more accurate estimators with larger sample sizes. This suggests that the algorithms will work well for data at the larger level like at national or state levels, but may not perform as well for samples from very small populations. As expected, our simulations also demonstrate that the algorithms work slightly better with a lower level of perturbation. We used the same stopping criterion regardless of the level of perturbation so it is possible that using using a stricter stopping criterion when we have a higher level of perturbation may lead to more significant bias adjustments.

It should be noted the algorithm did not perform as well on the 1993 Current Population Survey data across all the cases that were considered. One obvious reason is the range of underlying cell proportions, with π varying from 0.01 to 0.37. Another plausible explanation is the proposed algorithm assumes independence of covariates, which is not a realistic assumption in this example. An algorithm that allows for dependency between the covariates may be more applicable; we are currently exploring this issue and preliminary results are outlined in Woo (2013, Chapter 5). However, it should be also noted that the algorithm presented in this paper, despite ignoring some underlying associations, still provides significantly better results than when the unadjusted inference is done.

There seems to be a trend with the biasness of the slope in the regression models fitted to the data subject to PRAM. In all cases, the estimates for the slope had negative bias when the slope was positive, and positive bias when the slope was negative. This trend suggests that we have regression dilution in our setting, which has been established in the measurement error literature when a single continuous covariate is subject to measurement error. In the measurement error literature, however, there does not seem to be a general characterisation of direction and magnitude of the bias in a multivariate setting or when the response variable is subject to measurement error. In our analysis, however, we observe attenuation even when the response variable was subject to PRAM, and when multiple covariates were subject to PRAM. A more careful analysis needs to be carried out on the direction of bias, and how perturbation of both the response variable and the covariates affects the direction of bias, and to consider if such findings can improve the proposed algorithm.

Our results also show that with the application of PRAM, the estimates obtained using the proposed algorithms have higher standard errors when compared to the estimates obtained from the unadjusted inference. However, as sample size increases, the use of our algorithms produce smaller MSEs. This is mostly due to the significant reduction in bias as sample size increases. More careful analyses of the variance trends need to be done; e.g., How much of the variance is due to PRAM and how much is due to the algorithm? A more thoughtful consideration may need to be given if we are willing to accept the smaller bias at the expense of variance. Another consideration may be to find a PRAM matrix that minimises variance subject to a pre-specified level of bias that can be tolerated.

The main focus of our paper was on preserving data utility of microdata subject to PRAM when fitting a GLM. However, given our methodology, the weights from the proposed EM algorithm can also be used to estimate the disclosure risk, which is the probability that the observed values in the data after PRAM is applied is the same as the true values in the data before PRAM is applied. This type of risk measure makes use of the estimated regression coefficients. We are not aware of such a proposal in the current literature and this opens up new avenues of research. Since this potentially new measure of disclosure risk uses more information via regression estimates, we anticipate that it would be more accurate measure of risk. This however, requires further careful investigation.

Recent work by Shlomo and Skinner (2010) claims that combining sampling with a perturbation method like PRAM offers greater protection than using either method on its own. The same authors have also pointed out that PRAM itself may guarantee ε-differential privacy (Dwork, 2006), as long as the PRAM matrix does not contain zero elements. An interesting study would be to evaluate performance of PRAM only, PRAM with sampling and our EM methodology with respect to data utility. Depending on the results of the evaluations, we may need to consider adjusting the proposed EM algorithms to account for sampling. Similarly, the adjustments could be considered to ensure that the methodology satisfies the definition of ε-differential privacy, or possibly the more relaxed version of (ε,δ)-differential privacy.

There are other SDC methodologies that can be applied to categorical variables in microdata. Data swapping is used by agencies like the U.S. Census Bureau, and their approach guarantees that marginals involving the matching variables remain the same. However, the effect on regression analysis is ambiguous (Fienberg and McIntyre, 2004). Synthetic data methods are also becoming increasingly popular. Reiter (2005) carried out an empirical study using fully synthetic data with the 2000 Current Population Study, and found that the coverage probabilities for the logistic regression are extremely low. An interesting next step would be to compare the performance of synthetic data methodology to our proposed EM algorithms for PRAM.

Finally, it should be noted that in this paper we have worked with complete data. A next step would be to expand our methodology to also include missing data, and evaluate the impact of missingness in data on both the data utility and the disclosure risk in this setting.

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APPENDICES

A. Weights for EM Algorithm I

$$
\begin{aligned}
q_j &= P(W = w_j|W^* = w_k, Y, Z, \phi) \\
&= \frac{P(Y, W = w_j, W^* = w_k|Z, \phi)}{P(Y, W^* = w_k|Z, \phi} \\
&= \frac{P(Y|W = w_j, W^* = w_k, Z, \phi)P(W = w_j, W^* = w_k, Z, \phi)}{\sum_{l=1}^{J} P(Y, W^* = w_k, W = w_l|Z, \phi)} \\
&= \frac{P(Y|W = w_j, Z, \phi)P(W^* = w_k|W = w_j)P(W = w_j)}{\sum_{l=1}^{J} P(Y|W = w_l, W^* = w_k, Z, \phi)P(W^* = w_k|W = w_l)P(W = w_l)} \\
&= \frac{P(Y|w_j, Z, \phi)p_{W_{jk}}\pi_j}{\sum_{l=1}^{J} P(Y|w_l, Z, \phi)p_{W_{lk}}\pi_l}
\end{aligned}
$$

Note:
$P(Y|W, W^*, Z, \phi) = P(Y|W, Z, \phi)$ since when W is known, Y does not depend on W^*.
$P(W^*|W, Z, \phi) = P(W^*|W)$ because of PRAM, W^* only depends on W.
Also assume W, Z independent.

B. Weights for EM Algorithm II

$$
\begin{aligned}
r_j &= P(Y = j|Y^* = k, X, \beta) \\
&= \frac{P(Y = j, Y^* = k|X, \beta)}{P(Y^* = k|X, \beta)} \\
&= \frac{P(Y = j|X, \beta)P(Y^* = k|Y = j)}{\sum_l^J P(Y^* = k, Y = l|X, \beta)} \\
&= \frac{P(Y = j|X, \beta)p_{Y_{jk}}}{\sum_l P(Y^* = k|Y = l)P(Y = l|X, \beta)} \\
&= \frac{P(Y = j|X, \beta)p_{Y_{jk}}}{\sum_l p_{Y_{lk}}P(Y = l|X, \beta)}
\end{aligned}
$$

C. Weights for EM Algorithm III

$$
\begin{aligned}
s_{ml} &= P(Y=m, W=w_l | Y^*=k, W^*=w_j, Z, \phi) \\
&= P(Y=m | W=w_l, Y^*=k, W^*=w_j, Z, \phi) \cdot P(W=w_l | Y^*=k, W^*=w_j, Z, \phi) \quad .
\end{aligned}
$$

The first part is

$$
\begin{aligned}
&P(Y=m | W=w_l, Y^*=k, W^*=w_j, Z, \phi) \\
&= \frac{P(Y=m, W=w_l, Y^*=k, W^*=w_j | Z, \phi)}{P(W=w_l, Y^*=k, W^*=w_j | Z, \phi)} \\
&= \frac{P(Y^*=k | Y=m) P(Y=m | W=w_l, Z, \phi) P(W^*=w_j | W=w_l) P(W=w_l)}{\sum_a P(Y=a, W=w_l, Y^*=k, W^*=w_j | Z, \phi)} \\
&= \frac{P_{Y_{mk}} P(Y=m | W=w_l, Z, \phi) P(W^*=w_j | W=w_l) P(W=w_l)}{\sum_a P(Y^*=k | Y=a) P(Y=a | W=w_l, Z, \phi) P(W^*=w_j | W=w_l) P(W=w_l)} \\
&= \frac{P_{Y_{mk}} P(Y=m | W=w_l, Z, \phi)}{\sum_a P_{Y_{ak}} P(Y=a | W=w_l, Z, \phi)} \quad .
\end{aligned}
$$

The second part is

$$
\begin{aligned}
&P(W=w_l | Y^*=k, W^*=w_j, Z, \phi) \\
&= \frac{P(W=w_l, Y^*=k, W^*=w_j | Z, \phi)}{P(Y^*=k, W^*=w_j | Z, \phi)} \\
&= \frac{\sum_b P(Y=b, W=w_l, Y^*=k, W^*=w_j | Z, \phi)}{\sum_c \sum_d P(Y=d, W=w_c, Y^*=k, W^*=w_j | Z, \phi)} \\
&= \frac{\sum_b P(Y^*=k | Y=b) P(Y=b | W=w_l, Z, \phi) P(W^*=w_j | W=w_l) P(W=w_l)}{\sum_c \sum_d P(Y^*=k | Y=d) P(Y=d | W=w_c, Z, \phi) P(W^*=w_j | W=w_c) P(W=w_c)} \\
&= \frac{p_{W_{lj}} \pi_l \sum_b p_{Ybk} P(Y=b | W=w_l, Z, \phi)}{\sum_c p_{W_{cj}} \pi_c \sum_d p_{Ydk} P(Y=d | W=w_c, Z, \phi)} \quad .
\end{aligned}
$$

Hence

$$
\begin{aligned}
&P(Y=m, W=w_l | Y^*=k, W^*=w_j, Z, \phi) \\
&= \frac{p_{Y_{mk}} P(Y=m | W=w_l, Z, \phi)}{\sum_a p_{Y_{ak}} P(Y=a | W=w_l, Z, \phi)} \cdot \frac{p_{W_{lj}} \pi_l \sum_b p_{Ybk} P(Y=b | W=w_l, Z, \phi)}{\sum_c p_{W_{cj}} \pi_c \sum_d p_{Ydk} P(Y=d | W=w_c, Z, \phi)} \quad .
\end{aligned}
$$

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