# PRODUCTIVE SYSTEMS PROCESS MONITORING THROUGH PROCESS CAPABILITY INDICES [1]

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***Summary***

*Process capability indices are widely used in process assessments and in evaluation of decisions for a continuous improvement of quality in manufacturing or service process. This paper provides a brief survey of the main capability indices. The paper also focalizes the process capability indices in order to determine their sensitivity to time changes of the production system, showing some models, which emphasize both structural and random variables in the short or middle-long period of time. Then, we address the choice of suitable indices to make structural changes relying on the data process monitoring "ex post" or, alternatively, in real time when location or dispersion shifts from the target values are pointed out. Models are given and a Monte Carlo study has been perfomed.*

*Keywords: Univariate Process Capability Indices, Process Performance, Process Capability Indices Estimation.*

## 1. INTRODUCTION

Process capability indices are very important to control the production process and to point out what it should be done to keep it well. To this end, one has to get many data about the whole production system that, nowadays, are gained rather in real time through suitable statistical methods. The Total Quality Management (TQM), in particular, allows to connect all the different stages and processes

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in the production system and to respect the best conditions both of the design characteristics and of the customer satisfaction (Magagnoli, 1996). It is also necessary to refer to the international standards ISO-9000 (Lin, Chang, 2006).

There exist many studies about the employment of Process Capability Indices (PCI) which propose new indices in order to improve previously proposed ones and satisfy new requirements. Kotz, Johnson (1993 and 2002) and Kotz, Lovelace (1998) addressed estimation and hypothesis testing concerning PCI. Starting from the initial proposal of Juran *et alii* (1974), a wide spreading of PCI has been supported by a continuous use of them in the main international industries.

In this paper, we show the features of the main PCI, their use and the statistical properties of the suggested estimators in Sect. 2. In Sect. 3 we analyze their trend giving some models which point out both structural and random component due to variability in the short or middle-long period of time. The choice of the most useful indices to detect location or dispersion shifts from target values is addressed when the process is monitored "ex post" or real time. In Sect. 4 we analyze the trend of the suggested indices choosing the most frequently used ones which can give useful suggestions to monitor the productive process in order to describe "ex post" the process trend in terms both of dispersion values and of shifting from the target values. Results are mainly obtained through Monte Carlo methods and some comments are provided. Conclusion and future researches are discussed in Sect. 5.

## 2. MAIN PROCESS CAPABILITY INDICES

Let the examined characteristic $X$ be a one-dimensional random variable (r.v.), if we fix a *target T* and two values: *upper specification limit U* and *lower specification limit L* it is possible to value, as we compare them with the $X$ distribution, the proportion of the process output units which are not in conformity with the specified values

$$\theta = \theta_L + \theta_U = \Pr(X < L) + \Pr(X > U) = F_X(L) + (1 - F_X(U)) \tag{1}$$

where $F_X(\cdot)$ is the distribution function of $X$.

Another interesting parameter to evaluate the capability process is the mean loss or "risk" related to the $T$ value: $R = \mathrm{E}\{l(X; T)\}$, where $\mathrm{E}\{\bullet\}$ is the expected value and $l(x; T) \geq 0$ is the loss function which as been chosen according to the economic consequences given to the productive system or, more generally, to the Market System (purchase and sale) when the $x$ value of an output is not in conformity with $T$. According to the Taguchi methods about the robust design in TQM, it is frequently used a quadratic loss function $l(x; T) = a\,(x\text{-}T)^2$ that involves a risk function proportional to E $\{(X–T)^2\}$, so we can put, without losing generality

$$R \equiv \mathrm{E}\{(X\text{-}T)^2\} = \mathrm{E}\{(X\text{-}\mu)^2\} + (\mu\ T)^2 = \sigma^2 + (\mu - T)^2 \quad (2)$$

where $\mu$ and $\sigma^2$ are the mean and variance of $X$. In the light of this, it is necessary that the PCI are inversely related to the unconformity probability $\theta$ and the risk $R$, and therefore with the standard deviation $\sigma$ that measures the $X$ dispersion. Several other elements have also influence on $\theta$ and $R$, such as i) the distribution of $X$, in particular its shape (symmetrical or not; normal or not) and parameters related or not to location $\mu$ and standard deviation $\sigma$; ii) different specification settings: a unique specification value (upper or lower) instead of two (symmetric or asymmetric specification interval with respect to the target: $T = M$ or $T \neq M$, where $M = (U + L)/2$).

The typical feature of PCI is emphasized by considering the first proposed index in the literature:

$$C_p = (U - L)/6\sigma = d/3\sigma \quad (3)$$

which is defined by the ratio of two homogeneous intervals on $X$: the specification interval $(L, U)$ width, with $U - L = 2d$ and 6 times the standard deviation, that can be seen as the interval centered on E($X$) at which 0,27% of output units are not in conformity, assuming $X$ as normally distributed.

Starting from $C_p$, other PCI have been proposed in order to extend the $C_p$ idea (see Kotz, Lovelace (1998), 95). The most interesting of these extensions is the $C_p(u, v)$ due to Vännman (1995) and defined as

$$C_p(u,v) = \frac{d - u\,|\,\mu - M\,|}{3[\sigma^2 + v(\mu - T)^2]^{1/2}} \quad (4)$$

where $u$ and $v$ are non negative parameters. By setting alternatively, $u = 0$, 1 and $v = 0$, 1, $C_p(u, v)$ gives rise to the most used PCI.

For $u = v = 0$, we have

$$C_p(0, 0) = C_p. \quad (5)$$

As a consequence of the index definition, in the presence of a process distribution mean shift the $C_p$ index value does not change, hence it gives only the measure of the “possibility” that a productive process gets acceptable outputs (see Kotz, Lovelace (1998), 34-35). Other limits of the $C_p$ index are related to the assumptions one has to assume: i) $X$ normally distributed with mean $\mu$ and variance $\sigma^2$; ii) independent observations.

For $u = 1$ and $v = 0$, we have

$$C_p(1,0) = C_{pk} = \frac{\min(U-\mu, \mu-L)}{3\sigma} = \frac{d - |\mu - M|}{3\sigma} = \left[1 - \frac{|\mu - M|}{d}\right] C_p \,. \qquad (6)$$

This index takes into account the influence of the process mean value $\mu$, that is related to the defective output probability $\theta$. It is worth noting that $C_{pk} \le C_p$, where the sign of equality is valid for $\mu = M$.

For $u = 0$ and $v = 1$, we have

$$C_p(0,1) = C_{pm} = \frac{(U-L)}{6\tau} = \frac{d}{3[\sigma^2 + (\mu - T)^2]^{1/2}} \,. \qquad (7)$$

It is important to note that this index is related to the quadratic loss function approach. So this index does not only takes into account the process variability but also how much the process mean shifts from the target value. To maximize the index it suffices to minimize $\tau$, by reducing the process variability and/or by making the process mean close to the target value (see. Kotz, Lovelace (1998), 77-83).

For $u = v = 1$, we have

$$C_p(1,1) = C_{pmk} = \frac{\min(U-\mu, \mu-L)}{3\tau} = \left[1 - \frac{|\mu - M|}{d}\right] C_{pm} \,. \qquad (8)$$

This index, suggested by Pearn *et al.* (1992), gathers the properties of the other indices and is more sensitive to the process deviations from the target value.

This overview emphasizes the wealth of methodological indications that an inferential study on PCI can give. It should be also noted that sensitivity, efficiency and reliability of the PCI is deeply related to the practical needs of process control. A further contribution of the PCI studies is given by the links to the *Customer Satisfaction* and their use in TQM (see Magagnoli, Chiodini, Zappa (2002)).

## 3. THE PCI TREND: SOME METHODOLOGICAL ASPECTS

Let $X$ be a quantitative characteristic of the production process and $\Delta t = 1$ a time interval. We are interested in observing $X$ in the time interval $(0, s)$. The observed interval of length equal to $s$ is divided into $p$ subsequent periods ($k=1,\ldots,p$) with time width $m$, supposed constant, so that $s = p \times m$. Those subsequent periods can be marked by different real conditions, coming from the productive process performance.

For every time period we draw $m$ points of observation $j = 1,\ldots,m$, so that we have one observation for each $\Delta t$ unit interval, where $t = (k-1)m + j$ for $t = 1,\ldots,s$

with $k$=1,…,$p$ and $j$=1,…,$m$. Let us suppose to have, for every moment, $n_t=n$ observations $i$=1,…,$n$) (see Pearn e Yang, 2003).

A suitable model can be referred to a linear structure with random components, under the assumption of normality and independence. Those assumptions are very often met in practice. Zhang (1998) generalizes this framework so that

$$x_{kji} = \mu + \alpha_k + \beta_{kj} + \varepsilon_{kji}, \text{ for } k = 1,\dots,p,\ j = 1,\dots,m \text{ and } i = 1,\dots,n \tag{9}$$

where $\mu$ is the structural $X$ component that marks that the process is in mean for all the interval (0, $s$), while the other symbols are independent r.v.: $\varepsilon_{kji} \sim N(0,\sigma_E^2)$ for every $k$, $j$ is the random component, with null mean and constant variance $\sigma_\varepsilon^2$; $\alpha_k$ is the proper effect of the $k^{th}$ period with $\alpha_k \sim N(0,\sigma_\alpha^2)$ distribution, $\beta_{ki}$ is the proper effect of the $t^{th}$ sub-period, $t=(k-1)m+j$ with $N(0,\sigma_\beta^2)$ distribution, respectively. It is necessary to add to these assumptions of additive effects and normal distribution of every component also the independence between the class elements and the classes themselves $\{\alpha_k\}$, $\{\beta_{kj}\}$ e $\{\varepsilon_{kji}\}$.

Starting from the suggested model, other models, that show the proper structural and random components at periods and sub periods of observations, may be considered

$$x_{kji} = \begin{cases} \mu_{kj} + \varepsilon_{kji} \text{ where } \mu_{kj} = \mu + \alpha_k + \beta_{kj}, \text{ for } k = 1,\dots,p \text{ and } j = 1,\dots,m & \text{model a)} \\ \mu_k + \varepsilon_{ki'} \text{ where } \mu_k = \mu + \alpha_k,\ \varepsilon_{ki'} = \beta_{kj} + \varepsilon_{kji}, \text{ for } i' = (j-1)n + i & \text{model b)} \\ \mu + \varepsilon_{i*} \text{ where } \varepsilon_{i*} = \alpha_k + \varepsilon_{ki'}, \text{ for } i^* = (k-1)mn + (j-1)n + i & \text{model c)} \end{cases} \tag{10}$$

being, in the model a) $\mu_{kj}$ the $X$ mean of the sub period $t=(k-1)m+j$; in the model b) $\mu_k$ the $X$ mean of the $k^{th}$ period where the r.v. $\varepsilon_{ki'}$ is $N\left(0,\sigma_{\varepsilon'}^2\right)$ distributed for every $k$, where $\sigma_{\varepsilon'}^2 = \sigma_\beta^2 + \sigma_\varepsilon^2$ and for $i' = 1,\dots,mn$; in the model c) the r.v. $\varepsilon_{j*}$ is $N\left(0,\sigma_{\varepsilon*}^2\right)$ distributed, where $\sigma_{\varepsilon*}^2 = \sigma_\alpha^2 + \sigma_{\varepsilon'}^2$ for $i^* = 1,\dots,pmn$.

If we consider the process in conditions of regularity, without arrangements for shifting in mean from the target value, it is possible to assume that $\mu$ is constant, $\mu = \mu_0$, for all the observed periods and to call “basic situation” this assumption.

The productive process trend may show non regularity in different ways, either for the trend of the mean ($\mu$) and for the dispersion ($\sigma_E$), and ($\sigma_A$ e $\sigma_B$). Hence it is necessary to define the PCI sensitivity at the following situations of systematic non regulation in mean:

*i)* a linear shift of the sub periods mean: $\mu_k = \mu_0 + M_\alpha (k-1)/p$ for $k = 1, 2, \ldots, p$ where $M_\alpha$ is a positive constant, that can be explained by the general model as an effect on the mean of the distribution of $\alpha_k \sim N\left(M_\alpha (k-1)/p, \sigma_\varepsilon^2\right)$, as it is given in Figure 1 a);

*ii)* a quadratic shift of the mean in every sub period *p*: $\mu_{kj} = \mu_0 + M_\beta \left[(j-1)/(m-1)\right]^2$ for $j = 1, 2, \ldots, m$ where $M_\beta$ is a positive constant, which can be explained by the general model as an effect on the mean of the distribution of $\beta_{kj} \sim N\left(M_\beta \left[(j-1)/(m-1)\right]^2, \sigma_\beta^2\right) \forall k$, as it is given in Figure 1 b).

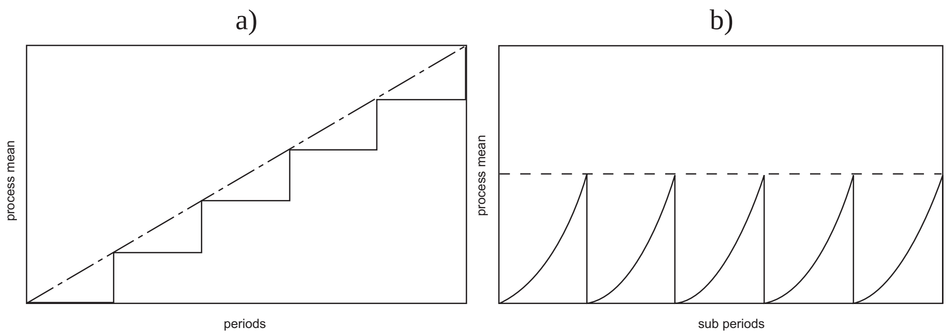


**Fig. 1: Non regular situations: a) linear shift of the period mean; b) quadratic shift of the mean in every sub period.**

It is possible to estimate the parameters of the considered models, following the analysis of variance methodology in presence of random “nested” effects (see Scheffé (1959), 248-255 and Lindman (1992), 127-142). The estimates of the means and variances, beginning from the sums of squares (*SS*) and their degrees of freedom (*g*), result as follows.

*Mean estimates*

$$\begin{cases} \hat{\mu}_{kj} = \sum_{i=1}^{n} x_{kji} \Big/ n, \text{ for } k = 1,\ldots,p \text{ and } j = 1,\ldots,m \\ \hat{\mu}_{k} = \sum_{j=1}^{m} \hat{\mu}_{kj} \Big/ m, \text{ for } k = 1,\ldots,p \\ \hat{\mu} = \sum_{k=1}^{p} \hat{\mu}_{k} \Big/ p \end{cases} \tag{11}$$

*Random component variance estimates*

$$SS_{\varepsilon(kj)} = \sum_{i=1}^{n} (x_{kji} - \hat{\mu}_{kj})^2, \qquad g_{\varepsilon(kj)} = n-1: \qquad \hat{\sigma}^2_{\varepsilon(kj)} = SS_{\varepsilon(kj)} / g_{\varepsilon(kj)},$$
$$\text{for } k = 1,\ldots p \text{ and } j = 1,\ldots,m \tag{12}$$

$$SS_{\varepsilon'(k)} = \sum_{j=1}^{m} \sum_{i=1}^{n} (x_{kji} - \hat{\mu}_{k})^2, \qquad g_{\varepsilon'(k)} = mn-1: \qquad \hat{\sigma}^2_{\varepsilon'(k)} = SS_{\varepsilon'(k)} / g_{\varepsilon'(k)},$$
$$\text{for } k = 1,\ldots,p \tag{13}$$

$$SS_{\varepsilon^*} = \sum_{k=1}^{p} \sum_{j=1}^{m} \sum_{i=1}^{n} (x_{kji} - \hat{\mu})^2, \qquad g_{\varepsilon^*} = pmn-1; \qquad \hat{\sigma}^2_{\varepsilon^*} = SS_{\varepsilon^*} / g_{\varepsilon^*} \tag{14}$$

Once the parameters required to calculate the PCI presented in Sect. 2 have been estimated, it is possible to estimate $C_p$, $C_{pk}$, $C_{pm}$ and $C_{pmk}$ by replacing $\mu$ and $\sigma$ with the estimates obtained in the different observations. The simulated observations were calculated though equation (9) summing up three independent random values $\alpha_k$, $\beta_{kj}$, $\varepsilon_{kji}$, as previously described. To work better by simulation procedures, we need to reduce the number of elements, even if the general structure and the consistency to real cases should be preserved. Since the different PCI, in equations (3), (6), (7) and (8), have the same components $\mu$ e $\sigma$, defining $C=C(\mu,\sigma)$ as a generic index, it is possible to calculate the following estimates with different structure and give some suggestions about the productive process behavior. Some expressions of the given PCI estimates, in different situations of means and variances in the studied model are presented

$$\begin{cases} \hat{C}_{\varepsilon(kj)} = C(\hat{\mu}_{kj}, \hat{\sigma}_{\varepsilon(kj)}), & \text{for } k = 1,\ldots,p \text{ and } j = 1,\ldots,m & \text{model a)} \\ \hat{C}_{\varepsilon'(k)} = C(\hat{\mu}_{k}, \hat{\sigma}_{\varepsilon'(k)}), & \text{for } k = 1,\ldots,p & \text{model b)} \\ \hat{C}_{\varepsilon^*} = C(\hat{\mu}, \hat{\sigma}_{\varepsilon^*}). & & \text{model c)} \end{cases} \tag{15}$$

where the mean and the standard deviation estimates are obtained by the equations (11) and (12)-(14).

To emphasize the PCI different behavior, we compare their estimates with measures of the defectiveness of the productive process trend. The defectiveness estimates of $\theta$ has been obtained through the mean and the variance values at the level of each unit interval of time $(j, k)$, of the sub periods $(k)$ and of the whole period

$$\begin{cases} \hat{\theta}_{kj} = \psi\left(\hat{\mu}_{kj}, \hat{\sigma}_{E(kj)}\right) & \text{for } k = 1, \cdots, p \;\text{ e }\; j = 1, \cdots, m \\ \hat{\theta}_{k} = \psi\left(\hat{\mu}_{k}, \hat{\sigma}_{E'(k)}\right) & \text{for } k = 1, \cdots, p \\ \hat{\theta} = \psi\left(\hat{\mu}, \hat{\sigma}_{E*}\right) & \end{cases} \tag{16}$$

where the equation $\psi\left(\overline{x}, s\right)$ is defined by (1) putting $\theta = F_X(L)+(1-F_X(U))= \psi(\mu, \sigma)$, assuming that $X$ is normal, being $\overline{x}$ and $s^2$ the mean and variance estimates, for a generic interval of time or sub period:

$$\psi\left(\overline{x}, s\right) = \Phi\left( (L - \overline{x})/s\right) + \Phi\left((\overline{x} - U)/s\right) \tag{17}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of the standardized normal.

In our opinion it is preferable to estimate the defectiveness by

$$\overline{\hat{\theta}}_k = \frac{1}{m}\sum_{j=i}^{m} \hat{\theta}_{kj}, \text{ for } k = 1, \ldots, p; \qquad \overline{\hat{\theta}} = \frac{1}{p}\sum_{k=1}^{p} \overline{\hat{\theta}}_k = \frac{1}{pm}\sum_{k=1}^{p}\sum_{j=1}^{m} \hat{\theta}_{kj} \tag{18}$$

respectively for a generic sub period and for the whole period. Regarding to the estimate values of the defectiveness $\left(\hat{\theta}_{kj}, \overline{\hat{\theta}}_k, \overline{\hat{\theta}}\right)$ we compare the corresponding values of the PCI estimates by mean of the corresponding relations given in the equation (15).

## 4. SIMULATION RESULTS

In order to evaluate the process performance and its PCI (either in a regular and in a non regular situation) and to draw general indications, a Monte Carlo study has been performed.

Assume, without loss of generality, the target value $T=M=0$, $\mu_0=0$ and $\sigma_\varepsilon=1$,

$L = -3$ and $U = 3$, $\Delta t$ equal to one hour, $n = 5$. The considered time period number is $p = 40$, every one made by a constant sub period number $m = 8$, so to have a whole period $s = p \cdot m = 40 \cdot 8 = 320$ hours. For every productive process situation, the simulation procedure has been repeated (in conditions of independence) for $r = 12$, hence we have $s \cdot r = 3840$ time units corresponding to a working year of two daily sub period for twenty days a month. Regarding to the mean standard deviations of the random components in model (10) we consider the following four *cases*.

We consider, as systematic process shifts, see Sect. 3, *i*) a linear deviation in mean of the random component $\{\alpha_k\}$ for every of the $p$ periods in the month: $\mu_\alpha(k) = M_\alpha(k-1)/p$ for $k$ 1, 2, …, $p$, with $M_\alpha = 1$; *ii*) a quadratic shift of the mean of the random component and hence: $\mu_\beta(j) = M_\beta\left[(j-1)/(m-1)\right]^2$ for $j = 1, 2, \ldots, m$, with $M_\beta \approx 0.5$. Then we study the three corresponding situations by considering: a situation "in control", one with "shift on $\{\alpha_k\}$ values", one with "shift on $\{\beta_{kj}\}$ values". Through Monte Carlo simulations we consider the four cases of Table 1. The $X$ values, the parameters (mean and variance) estimates and $C_p(u,v)$ with $u,v = 0$, 1 corresponding to the whole period have been computed. The procedure have been repeated for $r = 12$ successive periods (months).

**Tab. 1: Standard deviations of the model components in the considered cases**

| Cases | $\sigma_\alpha$ | $\sigma_\beta$ | $\sigma_{\varepsilon'} = \left[\sigma_\varepsilon^2 + \sigma_\beta^2\right]^{1/2}$ | $\sigma_{\varepsilon^*} = \left[\sigma_\alpha^2 + \sigma_{\varepsilon'}^2\right]^{1/2}$ |
|---|---|---|---|---|
| 1st | 0,25 | 0,1 | 1,0050 | 1,0356 |
| 2nd | 0,25 | 0,2 | 1,0198 | 1,0500 |
| 3th | 0,5 | 0,1 | 1,0050 | 1,1225 |
| 4th | 0,5 | 0,2 | 1,0198 | 1,1358 |

In order to compare the different PCI, whose empirical distributions are known by simulation, we compare the PCI estimates with the mean defectiveness ones $\{\theta(k, j)\}$ for every $(k, j)$ – *hour* –, with $\{\theta(k)\}$ of every sub period ($k$) and the one of all the observations – *month* or *year* – by mean of the equations (16) and (17) in Sect. 3. The main results are presented in Figures 2, 3, 4 that show how the structural changes of the model affect the PC indices.

Figure 2 reports the PCI and $\theta_{kj}$ estimates in the "basic situation" for what concern the first and fourth case reported in Table 1. Those cases are the two extreme situations regarding to the random components $\{\alpha_k\}$ and $\{\beta_{kj}\}$. The $\hat{\theta}_{kj}$ values are given in logarithmic scale and besides the graphic of the theoretic relation between the PCI and $\theta$ has been drawn when $\mu=0,\ \sigma_\varepsilon=1,\ \sigma_\alpha=0,\ \sigma_\beta=0$: ${}_M\theta(C)=2\Phi(-d\ C/\sigma_\varepsilon)=2\Phi(-3C)$, if the defectiveness is considered for $X$ values less than $L$ and for $X$ values upper than $U$, and: ${}_m\theta(C)=\Phi(-d\ C/\sigma_\varepsilon)=\Phi(-3C)={}_M\ \theta(C)/2$, if the only defectiveness is considered for $X$ values less than $L$ or upper than $U$. In particular the value of theoretical comparison of the fraction of defectiveness $\theta$ as function of the generic index $C=\dfrac{U-L}{6\sigma}=\dfrac{d}{3\sigma}$ is obtained considering the observable variable $X$ normally distributed with mean $\mu$ and variance $\sigma^2$ so that $\mu=T=M=\dfrac{U-L}{2}$, $\sigma=\sigma_\varepsilon$ and $d=3\sigma$. Under this conditions we obtain that, in the bilateral situation, ${}_M\theta$ is:

$${}_M\theta(C)=\Phi\left(\frac{L-\mu}{\sigma}\right)+\left[1-\Phi\left(\frac{U-\mu}{\sigma}\right)\right]=2\Phi\left(\frac{L-\mu}{\sigma}\right)=2\Phi\left(-\frac{(U-L)/2}{\sigma}\right)=2\Phi\left(-\frac{d}{\sigma}\right)=2\Phi\left(-3C\right)$$

instead, in the unilateral situation, is: ${}_m\theta(C)=\dfrac{{}_M\theta(C)}{2}=\Phi\left(-3C\right)$.

All the graphics in Figure 2 emphasize the gathering of the observations near to the graphic of the ${}_M\theta$ function; in particular, for $C_\mathrm{p}$ the estimated defectiveness are above the limit value ${}_M\theta$, while for $C_\mathrm{pk}$ about all values are between ${}_M\theta$ and ${}_m\theta$. $C_\mathrm{pm}$ and $C_\mathrm{pmk}$ seem more suitable since their estimated defectiveness values are less than ${}_M\theta(C)$, though the variability done by chance and by sampling. It should be also noted that $C_\mathrm{p}$ is determined by the only standard deviation and does not depend on the shifts of the mean estimated on the sample data, while the defectiveness estimate is affected. Since we noted a reduced sensitivity of the random components $\{\alpha_k\}$ and $\{\beta_{kj}\}$, in the comparison between the 1st *case* and the 4th *case* in Figure 2 , we focused on the 4th *case*, which shows a higher variability of those components.

Figure 3 reports PCI $\hat{C}_{E(kj)}$ and defectiveness $\hat{\theta}_{kj}$ monthly estimates for the non regular situations reported in Figure 1. In the comparison for each period we consider the estimates $\hat{C}_{E'(k)}=C(\hat{\mu}_k,\hat{\sigma}_{E'(k)})$, for $k$=1, …, $p$, defined by the equation (15) for model b).

To study the defective output proportion, we consider both $\hat{\theta}_k$, defined in (16),

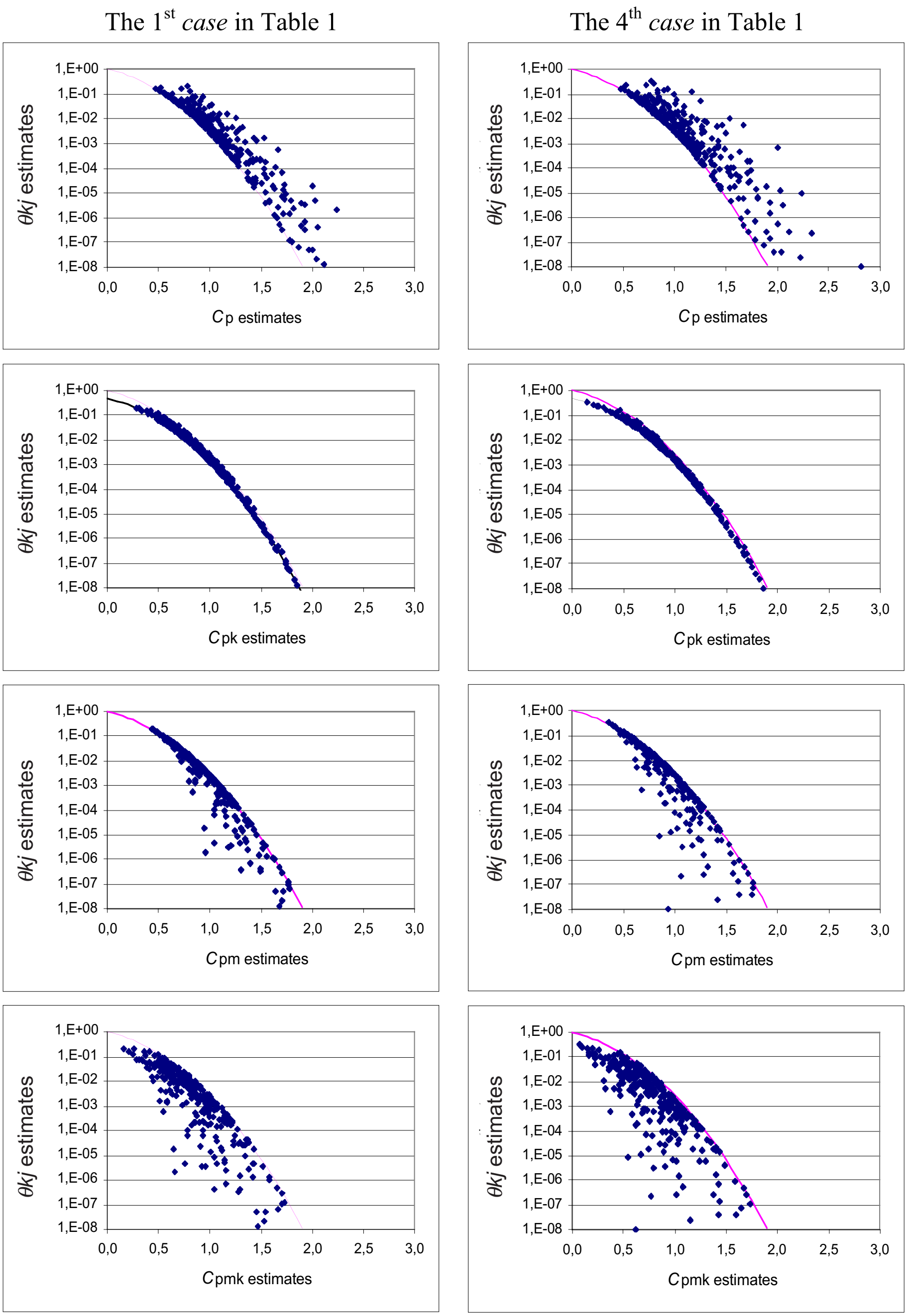


**Fig. 2: PCI and defective output proportion estimates graphics:** $\left\{\hat{C}_{\varepsilon(kj)} = C(\hat{\mu}_{kj}, \hat{\sigma}_{\varepsilon(kj)}); \hat{\theta}_{kj}\right\}$ **for *k*= 1, …, *p* = 40 and *j* = 1, …, *m* = 8 –"basic situation".**

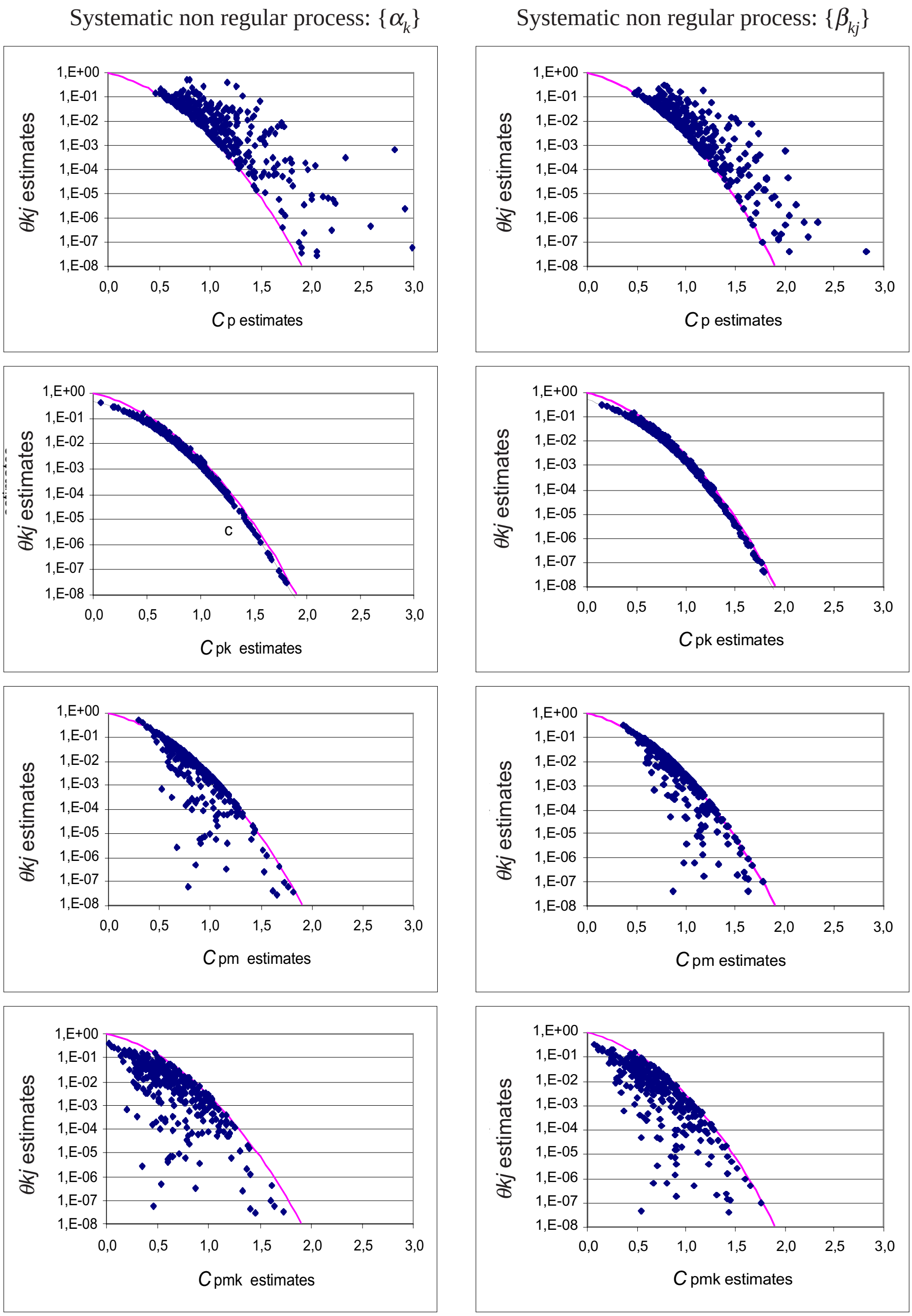


**Fig. 3: PCI and defective output proportion estimates graphics:** $\left\{\hat{C}_{\varepsilon(kj)};\hat{\theta}_{kj}\right\}$ **for** $k = 1, \ldots, p = 40$ **and** $j = 1, \ldots, m = 8$ **– "non regular situation" – "case 4th".**

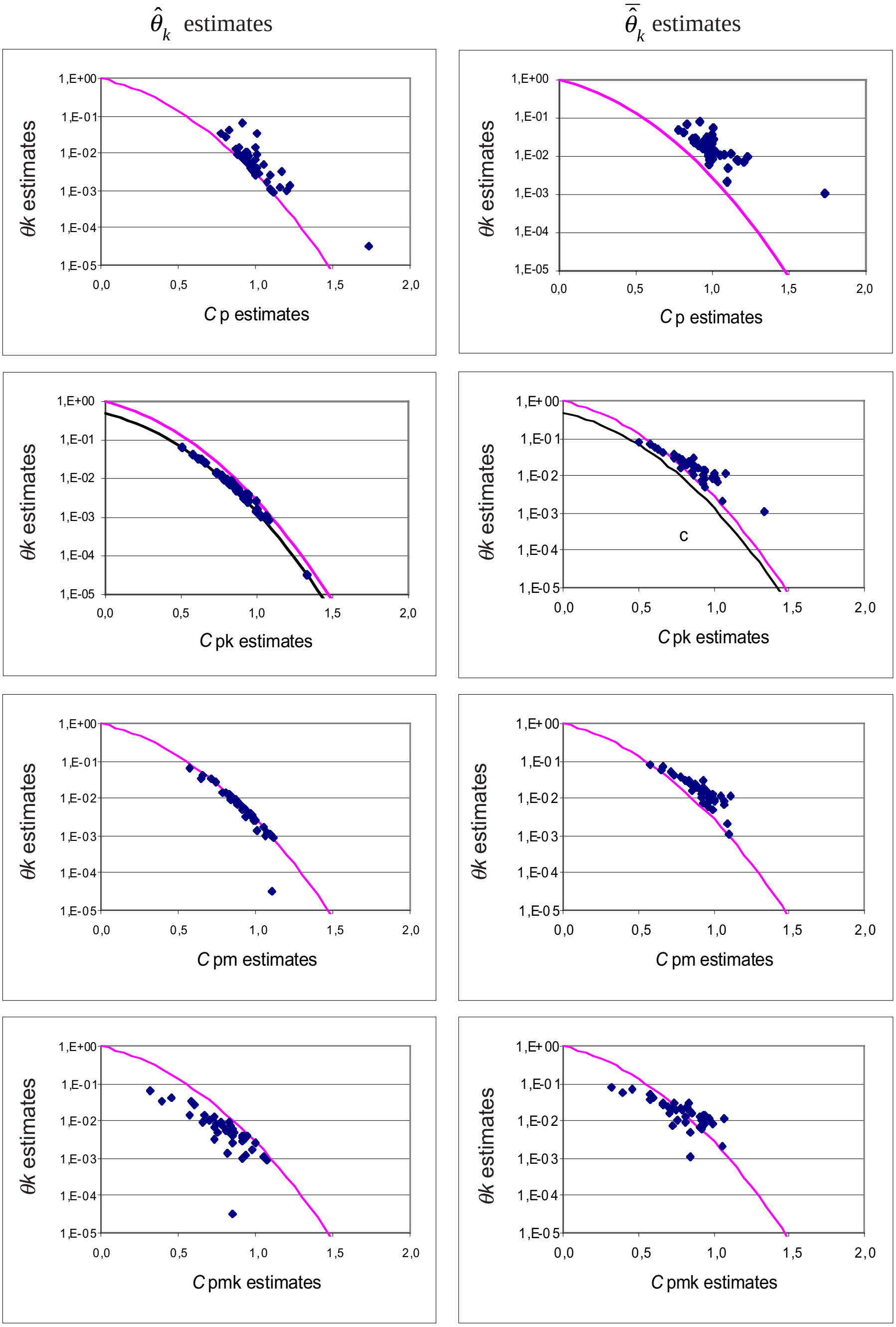


**Fig. 4: PCI and defective output proportion estimates graphics:**

$\left\{\hat{C}_{\varepsilon'(k)} = C(\hat{\mu}_{kj}, \hat{\sigma}_{\varepsilon'(k)}); \hat{\theta}_k \text{ or } \bar{\hat{\theta}}_k\right\}$ **for *k*= 1, …, *p* = 40 – "basic situation" – "4th case".**

and $\bar{\hat{\theta}}_k$, defined in (18). These estimates are graphically reported in Figure 4 ("basic situation", 4th *case*). The graphs in Figure 4 refer to $p = 40$. Note that the dispersion of the PCI values is lower than before, since this is a synthesis. If we consider the defectiveness estimates in terms of $\bar{\hat{\theta}}_k$, which we believe more coherent, $C_{\mathrm{pm}}$ or the more complex $C_{\mathrm{pmk}}$ index seem to be the most suitable to hold the different factors. We compare as well the estimates $\hat{C}_{E(kj)}$ with the defectiveness fraction estimates $\hat{\theta}_k$ and $\bar{\hat{\theta}}_k$, given in (16) and (18); the point corresponding to the $\hat{C}_{E(kj)}$ mean values and of their defectiveness fraction ($\hat{\theta}_k$ or $\bar{\hat{\theta}}_k$) has been plotted. The same conclusion can be derived by a theoretical analysis starting from the definitions of the suggested indices themselves, in particular studying the trend of $\bar{\hat{\theta}}_k$.

The frequency distribution trends of the estimated values $\hat{C}_{\varepsilon'(k)}$, $\hat{\theta}_k$ and $\bar{\hat{\theta}}_k$ are given in Figure 5, for the above considered data, there are $p \cdot r = 40 \cdot 12 = 480$ values. The PCI graphs have been obtained by classes with an interval of constant width equal to 0.15, while for the defectiveness proportion graph we consider the logarithmic transformation with classes of interval equal to 0.25. The distributions show a bell-shaped trend with a low emphasized asymmetry.

a) log $\hat{\theta}_k$ ; b) log $\bar{\hat{\theta}}_k$

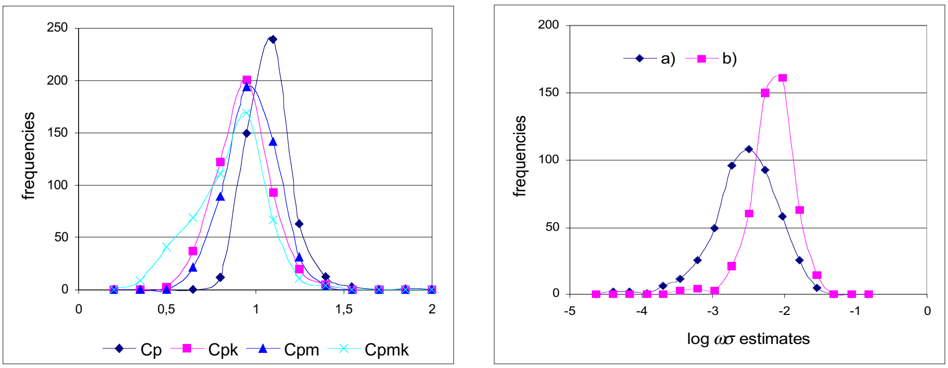


**Fig. 5: PCI and defective output proportion distribution estimates: $\left\{\hat{C}_{\varepsilon'(k)}; \hat{\theta}_k \text{ or } \bar{\hat{\theta}}_k\right\}$ for $k = 1, \ldots, p = 40$, extended to $r = 12$ – "4th case".**

## 5. FINAL REMARKS

Some suggestions can be drawn from the simulations that we carried out to study the behavior of a productive process trend when there are different structural and random factors which affect the *X* distribution. They confirm what is already well known about the studied and employed PCI.

Taking into account that every production process has to show regular conditions during the performance, from the results given in Sect. 4 it is possible to underline what follows.

- The use of $C_p$ index for observations at each unit of time or a period or a sub period gives indications of the random variability effect, and therefore of the capability process to respect the specification limit range that does not depend on the location of the mean production value in the observed period of time.
- The $C_{pk}$ index is more sensitive than the other PCI when data refer to unit intervals. Note that both variability and location, specified by the mean value act in its definition.
- If the productive system is considered at a sub period, the $C_{pk}$ and the $C_{pm}$ indices do not present evident differences, while the $C_{pmk}$ index is more tightly related to the presence of defective output.

Those considerations should be carefully examined by extending the considered models and parameter settings. However, it is worth noting that the planned software is simple to use and it is able to study the PCI behavior in many interesting applied circumstances.

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## REFERENCES

JURAN J.M., GRYNA F.M., BINGHAM R.S. jr (1974) *Quality Control Handbook*, McGraw-Hill, New York.

KOTZ S., JOHNSON N.L. (1993) *Process Capability Indices*, Chapman & Hall, London.

KOTZ S., JOHNSON N.L. (2002) Process Capability Indices - A Review, 1992-2000, with discussions, *Journal of Quality Technology*, 34, 1-53.

KOTZ S., LOVELACE C.R. (1998) *Process Capability Indices in Theory and Practice*, Arnold, London.

LIN C., CHANG S. (2006) Exploring TQM's Impact on the Casual Linkage between Manufacturing Objective and Organizational Performance, *Total Quality Management*, Vol. 17, 465-484.

LINDMAN H.R. (1992) *Analysis of Variance in Experimental Design*, Springer-Verlag, New York.

MAGAGNOLI U. (1996) Gli strumenti statistici nell'approccio della qualità totale, *Atti della XXXVIII Riunione Scientifica SIS*, Maggioli Editore, Rimini, Vol. 1, 249-260.

MAGAGNOLI U., CHIODINI P.M., ZAPPA D. (2002) Misure statistiche della capacità del processo: indicatori univariati e multivariati, *Atti della Riunione Satellite della XLI Riunione Scientifica SIS*, Cleup, Padova, 71-94.

PEARN W.L., YANG Y.S. (2003) Distributional and inferential properties of the estimated precision $C_p$ based on multiple samples, *Quality and Quantity*, 37, 443-453.

SCHEFFÉ H. (1959) *Analysis of Variance*, Wiley, New York.

VÄNNMAN K. (1995) A unified approach to capability indices, *Statistica Sinica*, 6, 805-820.

ZHANG N.F. (1998) Estimating process capability indexes for autocorrelated data, *Journal of Applied Statistics*, 25, 559-574.

## MONITORAGGIO DEI SISTEMI PRODUTTIVI MEDIANTE INDICI DI CAPACITÀ DI PROCESSO

***Sommario***

*Gli indici di capacità di processo sono diffusamente impiegati per la valutazione di processi sia produttivi che aziendali e in particolare nell'assumere decisioni al fine di migliorare la qualità della produzione di beni o l'erogazione di servizi. Il presente contributo fornisce una breve rassegna dei principali indici di capacità e si focalizza a considerare i problemi di stima di tali indici e la loro diversa sensibilità nell'evidenziare le condizioni di funzionamento del processo che possono manifestarsi durante la produzione. Vengono presentati alcuni modelli in grado di seguire l'evoluzione temporale del processo nel breve e nel medio-lungo termine. Sono date indicazioni per la scelta, tra gli indici di processo considerati, di quelli che possono meglio monitorare il processo in presenza di scostamenti, in termini di locazione e di dispersione rispetto ai valori di target, che possono attuarsi. Lo studio è stato condotto mediante simulazioni delle differenti situazioni e casi operativi considerati, sia di regolarità che di sregolazione, avvalendosi di procedure numeriche Monte Carlo.*