THE DECOMPOSITION OF THE ATKINSON-PLOTNICK-KAKWANI RE-RANKING MEASURE

Maria Giovanna Monti, Achille Vernizzi

Department of Economics, Management and Quantitative Methods, Università degli Studi di Milano, Milan, Italy

Mauro Mussini[1]

Department of Statistics, Università degli Studi di Milano "Bicocca", Milan, Italy

Abstract *This article introduces a new matrix approach, based on pairwise income comparisons, to decompose the Atkinson-Plotnick-Kakwani re-ranking measure when income units are gathered into groups. As it is known the Atkinson-Plotnick-Kakwani re-ranking measure is defined by the difference between the Gini index and the concentration index. Re-ranking can be exclusively decomposed as the sum of two parts: the within group and the across-group re-ranking measures. Our results are applied to a household sub-sample selected from the 2007 survey of the Panel Study on Income Dynamics (PSID). In the applied analysis our approach proves to be quite powerful. We show that in addition to providing different measures, it allows to calculate the incidence of re-ranking.*

Keywords*: Gini index, Concentration index, Re-ranking, Index decompositions, Population groups.*

1. INTRODUCTION

A tax system induces re-ranking when the rank ordering of after-tax incomes is different from that of before-tax incomes. Re-ranking between income receivers can be considered from different standpoints. For instance, some can see re-ranking positively in that it points to income mobility. As noted by Wagstaff (2009), re-ranking might be argued to point to the existence of equality of opportunity, at least if it reflects the results of different efforts rather than differential endowments. On the other hand, from a different perspective, re-ranking may be seen as inequitable.

[1] Maria Giovanna Monti, email: mariagiovanna.monti@unimib.it;
Mauro Mussini, email: mauro.mussini1@unimib.it;
Achille Vernizzi, email: achille.vernizzi@unimi.it

In income tax literature, it is argued that procedural fairness requires that tax system does not alter the ranking of income receivers in the move from before-tax to after-tax income distribution (see among others: Feldstein, 1976; Atkinson, 1980; Plotnick, 1981). This paper considers the re-ranking measurement as well as its decomposition by group in the latter frame.

The re-ranking effect of taxation has been isolated through the distinct contributions provided by Atkinson (1980), Plotnick (1981) and Kakwani (1984). The measure of the re-ranking effect is known as the Atkinson-Plotnick-Kakwani (henceforth, APK) re-ranking index and it is given by the difference between the after-tax Gini index and the concentration index for the after-tax incomes sorted by the respective before-tax incomes. Since its introduction, a number of studies have been published dealing with the calculation of the APK re-ranking index (among others: Jenkins, 1988; Aronson and Lambert, 1994; Van de Ven *et al.*, 2001), however, less attention has been paid to the group decomposition of the index. Recently, Urban and Lambert (2008) (henceforth U.L.) proposed a decomposition of the APK re-ranking index considering income receipts partitioned on the basis of their before-tax incomes and contiguous income classes. By definition, in the U.L. re-ranking decomposition classes do not overlap before taxes. Nevertheless, situations where before-tax incomes overlap among groups are very common in several contexts (e.g., groups formed by household type, age, geographical area, gender, etc.). In a sense, this paper generalises the U.L. analysis by considering groups which may overlap before and after taxes.

For this purpose, we measure the overall re-ranking adopting an approach based on pairwise income comparisons between all pairs of income receipts. In so doing, we show that the APK re-ranking measure is decomposable as the sum of two terms: a within-group re-ranking measure and an across-group re-ranking measure which reckons re-ranking between all pairs of income receipts belonging to different groups. The across-group re-ranking component can be further split into two components: the between-group re-ranking component and a component which is linked to the overlapping existing before and after taxation. However, we will see that these two components are unsuitable for explaining the re-ranking effect of taxation when considering re-ranking among income receivers (e.g., individuals, households). As a consequence, considering income units gathered into groups, we can state that re-ranking is decomposable as a sum of two parts: the measure of re-ranking within groups and the measure of re-ranking across groups.

To obtain these results we provide a matrix expression for the APK re-ranking index. We start by developing matrix expressions for the Gini and concentration indices. The literature offers a number of alternative approaches to calculate and to decompose the Gini index (see Giorgi, 2011). In particular, Pyatt (1976), Silber (1989) and Yao (1999) endorse matrix form approaches. Differently from the Gini index, the concentration index is usually expressed as a function of the underlying concentration curve. We suggest an alternative expression for the concentration index and a new matrix form approach that can be applied to both indices. This matrix approach is appealing since it yields comparable expressions for the two indices, so that the Gini index and the concentration index can be decomposed following the same procedure. We remark that the overall indices and their components are expressed as functions of the same matrix.

The presented decompositions are then used to decompose the APK re-ranking measure. By first, we split re-ranking as a sum of re-ranking within and across groups. It is simple to see that these two components are both re-ranking measures. Then, following the suggestion of the Gini index decomposition where the across-group component splits into the between and overlapping components (Dagum, 1997), we try to decompose the re-ranking across groups into two terms. The expressions we use to obtain the across-group re-ranking decomposition allow to show that these two components are not proper re-ranking measures when re-ranking between individuals is measured. It follows that, differently from the Gini index, the re-ranking can be partitioned only as the sum of two re-ranking components.

We apply the obtained results to a household sub-sample selected from the 2007 survey of the Panel Study on Income Dynamics (PSID). The survey provides income, employment and demographic data for individuals and households included in a representative sample of the United States population (PSID, 2012). From this sample 3,505 families are selected and partitioned into three groups, each group being characterized by different number of children.

The paper is organized as follows. In Section 2, we present the matrix form for the Gini and concentration indices; then, the decompositions of the two indices are obtained. In Section 3, we present the matrix form of the re-ranking index R^{APK} and discuss its decomposition. In Section 4, we apply the decomposition to real data. Section 5 concludes.

2. GINI AND CONCENTRATION INDEX IN MATRIX FORM

Let X and Y be the before-tax (hereafter, *b.t.*) and the after-tax (hereafter *a.t.*) income distribution for a population of N individuals, respectively. Let x_i and y_i be the *b.t.* and the *a.t.* incomes of the individual i, respectively. Let p_i be the weight associated to the pair (x_i, y_i), with $\sum_{i=1}^{N} p_i = N$. When the N individuals are sampled by a complex sample design, p_i is the sampling weight for the pair (x_i, y_i). In the simplest case, every p_i is equal to 1. The X-ordering denotes the ordering of the (x_i, y_i, p_i) sequence when all elements are lined up by the non-decreasing ordering of X. The Y-ordering denotes the ordering of the (x_i, y_i, p_i) sequence when all elements are sorted by the non-decreasing ordering of Y. Let $r_Y(y_i)$ and $r_Y(x_i)$ denote the ranks of y_i and x_i in the Y-ordering, respectively; analogously $r_X(y_i)$ and $r_X(x_i)$ are the ranks of y_i and x_i in the X-ordering.

To derive the matrix form for the Gini and concentration indices we start defining the Gini index, G_Y, in terms of Gini Mean Difference as

$$G_Y = \frac{1}{2\mu_Y N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(y_i - y_j\right) p_i p_j \cdot I\left\{r_Y\left(y_i\right) - r_Y\left(y_j\right)\right\} \tag{1}$$

where μ_Y denotes the *a.t.* weighted average income and $I\{z\}$ is an indicator function[2] that equals 1 if $z \geq 0$ and -1 if $z < 0$.[3]

Then, one remembers that the concentration index for a variable with respect to another can be calculated lining up the values of the first variable by the increasing ordering of the second (Lambert, 2001). Thus, the concentration index for *a.t.* incomes with respect to *b.t.* incomes, $C_{Y|X}$, defines as

$$C_{Y|X} = \frac{1}{2\mu_Y N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(y_i - y_j\right) p_i p_j \cdot I\left\{r_X\left(y_i\right) - r_X\left(y_j\right)\right\}. \tag{2}$$

Let us introduce the following notation:

$\mathbf{y}$ is the $N\times 1$ vector where *a.t.* incomes are stacked according to the Y-ordering;

$\mathbf{y}_X$ is the $N\times 1$ vector where *a.t.* incomes are stacked according to the X-ordering;

$\mathbf{p}_Y$ is the $N\times 1$ vector containing weights stacked as the elements of $\mathbf{y}$;

$\mathbf{p}_X$ is the $N\times 1$ vector containing weights stacked as the elements of $\mathbf{y}_X$;

$\mathbf{E}$ is a $N\times N$ permutation matrix,[4] such that

$\mathbf{p}_X = \mathbf{E}\mathbf{p}_Y$; $\mathbf{E}'\mathbf{p}_X = \mathbf{p}_Y$; $\mathbf{y}_X = \mathbf{E}\mathbf{y}$; $\mathbf{E}'\mathbf{y}_X = \mathbf{y}$;

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2 On the indicator function, see Faliva (2000).

3 When two individuals have the same after-tax income value, their relative positions in the after-tax ranking remain as in the before-tax income parade. By construction two different individuals cannot occupy a same rank position even if their incomes are equal.

4 Observe that $\mathbf{E}^{-1} = \mathbf{E}'$; for definitions concerning permutation matrices, see Faliva (1996).

S denotes a $N\times N$ antisymmetric matrix with diagonal elements equal to zero, super-diagonal elements equal to 1 and sub-diagonal elements equal to -1;

j is a $N\times 1$ vector with entries equal to 1;

$\mathbf{D}_Y$ and $\mathbf{D}_{Y|X}$ denote the $N\times N$ matrices $\mathbf{D}_Y = \left(\mathbf{jy}' - \mathbf{yj}'\right)$ and $\mathbf{D}_{Y/X} = \left(\mathbf{jy}'_X - \mathbf{y}_X\mathbf{j}'\right)$.

Using the Hadamard product $\circ$ we rewrite expression (1) as

$$G_Y = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y{}'\left(\mathbf{S}\circ\mathbf{D}_Y\right)\mathbf{p}_Y . \tag{3}$$

Following the definition of the concentration index one has

$$C_{Y|X} = \frac{1}{2\mu_Y N^2}\mathbf{p}_X{}'\left(\mathbf{S}\circ\mathbf{D}_{Y|X}\right)\mathbf{p}_X , \tag{4}$$

then, observing that $\mathbf{E}'\mathbf{D}_{Y|X}\mathbf{E} = \left(\mathbf{jy}_X{}'\mathbf{E} - \mathbf{E}'\mathbf{y}_X\mathbf{j}'\right) = \mathbf{D}_Y$ and applying the Hadamard product properties, we rewrite (4) as (Vernizzi, 2009)

$$C_{Y|X} = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y{}'\left(\mathbf{E}'\mathbf{SE}\circ\mathbf{D}_Y\right)\mathbf{p}_Y . \tag{5}$$

Expression (5) is the matrix form for expression (2).

When a population is partitioned in *H* groups, equations (3) and (5) can be decomposed by group. Groups can be formed according to any criterion (e.g., socio-demographic characteristics, income class, geographical area, occupational attainment, etc.) and *b.t.* income ranges may overlap, as it occurs in various empirical applications.[5] From these perspectives, the decomposition approach we propose is very general. Equations (3) and (5) pave the way for a representation of Gini and concentration index components as functions of the same difference matrix $\mathbf{D}_Y$.

When considering a population split into *H* groups, it is useful to index the triplet as $(x_{h,l}, y_{h,l}, p_{h,l})$ where *h* is the group index (h=1, 2,...,H), and *l* is the individual index within group *h* (l=1, 2,..., N_h). The *a.t.* and *b.t* average incomes of

[5] Analysing the effects of tax reforms on income distribution, policy makers can split the population into groups formed by various criteria (household type, income source, employment type, age, etc.) to measure the tax impact on the income of a specific subpopulation.

the group *h* are denoted by μ_{Yh} and by μ_{Xh}, respectively. Following Dagum's (1997) Gini decomposition as discussed in Monti (2008), we decompose the index into a within-group component, G_Y^W, and an across-group component, G_Y^{AG}. These two terms can be written as

$$G_Y^W = \frac{1}{2\mu_Y N^2}\sum_{h=1}^{H}\left[\sum_{l=1}^{N_h}\sum_{m=1}^{N_h}\left(y_{h,l}-y_{h,m}\right)p_{h,l}p_{h,m}\cdot I\left\{r_Y\left(y_{h,l}\right)-r_Y\left(y_{h,m}\right)\right\}\right], \qquad (6)$$

$$G_Y^{AG} = \frac{1}{2\mu_Y N^2}\sum_{h=1}^{H}\sum_{g\neq h}^{H}\left[\sum_{l=1}^{N_h}\sum_{m=1}^{N_g}\left(y_{h,l}-y_{g,m}\right)p_{h,l}p_{g,m}\cdot I\left\{r_Y\left(y_{h,l}\right)-r_Y\left(y_{g,m}\right)\right\}\right]. \qquad (7)$$

Moreover, the across-group component can be decomposed into the sum of between component, G^B, and overlapping component, G^T. The overlapping component defines as in (8) once groups have been sorted and indexed by the increasing order of their average incomes,

$$G_Y^T = \frac{1}{\mu_Y N^2}\sum_{h=2}^{H}\sum_{g=1}^{h-1}\left[2\sum_{l=1}^{N_h}\sum_{m=1}^{N_g}\left(y_{g,m}-y_{h,l}\right)p_{h,l}p_{g,m}\right] \qquad y_{h,l} < y_{g,m}\,. \qquad (8)$$

The summation within brackets in (8) is the sum of the differences ($y_{g,m}$-$y_{h,l}$) between incomes belonging to all pairs of income receivers for which the inequalities $y_{h,l} < y_{g,m}$ and $\mu_{Yh} > \mu_{Yg}$ hold. Following Gini (1959) we recall that whenever a member of the poorer (on average) group is richer than a member of the richer (on average) group one has a *transvariation*. Then, using Gini's terminology the overlapping term in (8) defines as twice the weighted sum of (intensity of) transvariations. Given (8), one obtains the between-group component, G_Y^B, subtracting (8) from (7),

$$G_Y^B = \frac{1}{\mu_Y N^2}\sum_{h=2}^{H}\sum_{g=1}^{h-1}\left(\sum_{l=1}^{N_h}\sum_{m=1}^{N_g}\left(y_{h,l}-y_{g,m}\right)p_{h,l}p_{g,m}\right). \qquad (9)$$

It is known that the between-group component measures the disparities between group average incomes. However, we remark that expression (9) defines G_Y^B as a function of the differences between incomes instead of the differences between group average incomes.

Expressions (6) and (7) rewrite in compact matrix forms as in (10) and (11) by aligning incomes according to the *Y*-ordering,

$$G_Y^W = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left(\mathbf{W}_Y \circ \mathbf{S} \circ \mathbf{D}_Y\right)\mathbf{p}_Y, \tag{10}$$

$$G_Y^{AG} = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left[\left(\mathbf{J}-\mathbf{W}_Y\right)\circ \mathbf{S} \circ \mathbf{D}_Y\right]\mathbf{p}_Y. \tag{11}$$

In (10) and (11), $\mathbf{W}_Y$ is a $N\times N$ matrix with non-zero elements equal to 1 in the entries which correspond to the differences between incomes of a same group in $\mathbf{D}_Y$. $\mathbf{J}$ is a $N\times N$ matrix with all elements equal to 1. The Hadamard product $\mathbf{W}_Y \circ \mathbf{D}_Y$ selects the $\sum_{h=1}^{H} N_h^2$ pairwise differences between incomes of a same group in $\mathbf{D}_Y$, while $(\mathbf{J}-\mathbf{W}_Y)\circ\mathbf{D}_Y$ selects the $\left(N^2-\sum_{h=1}^{H} N_h^2\right)$ pairwise differences between incomes belonging to different groups in $\mathbf{D}_Y$.

One obtains the matrix form for G_Y^B introducing the permutation matrix[6] $\mathbf{A}_Y$,

$$G_Y^B = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left[\left(\mathbf{J}-\mathbf{W}_Y\right)\circ \mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y \circ \mathbf{D}_Y\right]\mathbf{p}_Y. \tag{12}$$

Subtracting (12) from (11), one has the matrix form of the overlapping component expressed as in equation (8),

$$G_Y^T = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left[\left(\mathbf{J}-\mathbf{W}_Y\right)\circ\left(\mathbf{S}-\mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y\right)\circ \mathbf{D}_Y\right]\mathbf{p}_Y. \tag{13}$$

The concentration index is decomposed by following the same approach used for G_Y. We first decompose this index into two components $C_{Y|X}^W$ and $C_{Y|X}^{AG}$, which are analogous to G_Y^W and G_Y^{AG}. One calculates $C_{Y|X}^W$ and $C_{Y|X}^{AG}$ by replacing *b.t.*

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6 In $\mathbf{A}_Y\mathbf{y}$, after-tax incomes are lined up in non-decreasing ordering within each group and groups follow the non-decreasing order of their mean. Given $\mu_{Yh} \le \mu_{Yh+1}$, one has $\left(\mathbf{A}_Y\mathbf{y}\right)' = \left[\left(y_{1,1}, y_{1,2}, \dots y_{1,N_1}\right), \dots, \left(y_{h,1}, y_{h,2}, \dots y_{h,N_h}\right), \dots, \left(y_{H,1}, y_{H,2}, \dots y_{H,N_H}\right)\right]$, $y_{h,l} \le y_{h,l+1}$.

incomes in the *b.t.* Gini index expression with the corresponding *a.t.* incomes.

After some algebraic manipulations, one obtains

$$C_{Y|X}^{W}=\frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left(\mathbf{W}_Y\circ\mathbf{E}'\mathbf{S}\mathbf{E}\circ\mathbf{D}_Y\right)\mathbf{p}_Y\,, \tag{14}$$

$$C_{Y|X}^{AG}=\frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left[\left(\mathbf{J}-\mathbf{W}_Y\right)\circ\mathbf{E}'\mathbf{S}\mathbf{E}\circ\mathbf{D}_Y\right]\mathbf{p}_Y\,. \tag{15}$$

Analogously to G_Y^{AG}, the across-group concentration index is decomposable as the sum of $C_{Y|X}^{B}$ and $C_{Y|X}^{T}$. These terms are calculated by replacing the *b.t.* incomes with the respective *a.t.* incomes in the between and overlapping components of the *b.t.* Gini index.

Now, introducing the permutation matrix $\mathbf{A}_X$,[7] one has

$$C_{Y|X}^{B}=\frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left[\left(\mathbf{J}-\mathbf{W}_Y\right)\circ\mathbf{E}'\left(\mathbf{A}_X'\mathbf{S}\mathbf{A}_X\right)\mathbf{E}\circ\mathbf{D}_Y\right]\mathbf{p}_Y\,. \tag{16}$$

Then, subtracting (16) from (15), one obtains the expression of $C_{Y|X}^{T}$,

$$C_{Y|X}^{T}=\frac{1}{2\mu_Y N^2}\mathbf{p}_Y'\left\{\left(\mathbf{J}-\mathbf{W}_Y\right)\circ\left[\mathbf{E}'\mathbf{S}\mathbf{E}-\mathbf{E}'\left(\mathbf{A}_X'\mathbf{S}\mathbf{A}_X\right)\mathbf{E}\right]\circ\mathbf{D}_Y\right\}\mathbf{p}_Y\,. \tag{17}$$

The Gini and concentration indices as well as their components are now written as a function of the same matrix of differences $\mathbf{D}_Y$.

3. THE RE-RANKING INDEX R^{APK} AND ITS DECOMPOSITION

A tax system induces re-ranking among income receivers when the *a.t.* ranking of income receivers differs from the *b.t.* one. The APK index measures re-ranking between ungrouped income receivers and it is equal to the difference between the *a.t.* Gini index and the concentration index for *a.t.* incomes calculated by keeping

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[7] $\left(\mathbf{A}_X\mathbf{x}\right)'=\left[\left(x_{1,1},x_{1,2},\ldots x_{1,N_1}\right),\ldots,\left(x_{h,1},x_{h,2},\ldots x_{h,N_h}\right),\ldots,\left(x_{H,1},x_{H,2},\ldots x_{H,N_H}\right)\right]$, $x_{h,l}\le x_{h,l+1}$ and $\mu_{Xh}\le\mu_{Xh+1}$.

income receivers arranged by non-decreasing order of their *b.t.* income. Using (3) and (5) we write the APK re-ranking index R^{APK} in matrix form as

$$R^{APK} = \frac{1}{2\mu_Y N^2} \mathbf{p}_Y{}' \left[\mathbf{S}^{APK} \circ \mathbf{D}_Y \right] \mathbf{p}_Y, \tag{18}$$

where $\mathbf{S}^{APK} = \mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E}$. Recalling that $\mathbf{S} = \left[s_{ij} \right]$ is the matrix of signs, we remark that $\mathbf{E}'\mathbf{S}\mathbf{E} = \left[s_{ij}^e \right]$ maps the permutation of each income from the *X*-ordering to the *Y*-ordering. To describe how (18) yields R^{APK} let us consider only the super-diagonal elements of $\mathbf{D}_Y = \left[d_{ij}^Y \right]$, $d_{ij}^Y = y_j - y_i \geq 0$, where the indexes *i* and *j* coincide with the ranks $r_Y(y_i)$ and $r_Y(y_j)$ of y_i and y_j in the *Y*-ordering. Then, for the pair (*i*,*j*) one has:

$s_{ij}^e = 1$, if $r_Y\left(y_j\right) > r_Y\left(y_i\right)$ and $r_X\left(y_j\right) > r_X\left(y_i\right)$.

$s_{ij}^e = -1$, if $r_Y\left(y_j\right) > r_Y\left(y_i\right)$ and $r_X\left(y_j\right) < r_X\left(y_i\right)$.

As a consequence, in (18) the elements $\left(s_{ij} - s_{ij}^e\right)$ are + 2 whenever a re-ranking occurs, otherwise they are zero.[8] So, as the super-diagonal elements can be associated to either 0 or 2, expression (18) confirms the well-known result $0 \leq R^{APK} \leq 2G_Y$.

When population is partitioned into groups, we can decompose R^{APK} by group using the $C_{Y|X}$ and G_Y components. As shown in the previous section all these terms can be expressed as function of $\mathbf{D}_Y$. We use expressions (10), (11), (14) and (15) to decompose R^{APK} as sum of within and across-group re-ranking components ($R^{APK} = R^{AG} + R^W$; $R^{AG} = G^{AG} - C^{AG}$; $R^W = G^W - C^W$).

The difference between (10) and (14) yields the measure of the re-ranking within groups,

$$R^W = \frac{1}{2\mu_Y N^2} \mathbf{p}_Y{}' \left[\mathbf{S}^W \circ \mathbf{D}_Y \right] \mathbf{p}_Y \tag{19}$$

where $\mathbf{S}^W = \mathbf{W}_Y \circ \left(\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E} \right)$. We observe that the contribution of the re-ranking occurring within group *h* to the overall re-ranking measure can be isolated by

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8 In case of re-ranking, the elements in the sub-diagonal part of $\mathbf{D}_Y$ $\left(d_{ij}^Y = y_j - y_i \leq 0\right)$ are multiplied by $\left(s_{ij} - s_{ij}^e\right) = -2$.

replacing $\mathbf{S}^W$ in equation (19) with $\mathbf{S}_h^W = \mathbf{W}_{Y,h} \circ \left(\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E}\right)$, where $\mathbf{W}_{Y,h}$ is a $N\times N$ matrix with non-zero elements equal to 1 in the entries which correspond to the differences between incomes of group h in $\mathbf{D}_Y$.

The difference between expressions (11) and (15) yields the measure of the re-ranking between individuals belonging to different groups that is the across-group re-ranking,

$$R^{AG} = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y{}'\left[\mathbf{S}^{AG} \circ \mathbf{D}_Y\right]\mathbf{p}_Y. \tag{20}$$

where $\mathbf{S}^{AG} = \left(\mathbf{J} - \mathbf{W}_Y\right)\circ\left(\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E}\right)$. More specifically, the contribution of the re-ranking occurring across groups h and g to the overall re-ranking measure can be isolated by replacing $\mathbf{S}^{AG}$ in (20) with $\mathbf{S}_{hg}^{AG} = \mathbf{W}_{Y,hg} \circ \left(\mathbf{S} - \mathbf{E}'\mathbf{S}\mathbf{E}\right)$, where $\mathbf{W}_{Y,hg}$ is a $N\times N$ matrix with non-zero elements equal to 1 in the entries which correspond to the differences between incomes of groups h and g in $\mathbf{D}_Y$.

According with the Gini index decomposition, the across-group re-ranking R^{AG} can be split into the sum of two terms: R^B and R^T (R^{AG}=R^B+ R^T; $R^B = G_Y^B - C_{Y|X}^B$; $R^T = G_Y^T - C_{Y|X}^T$). The term R^B evaluates the re-ranking between group means so that one has R^B= 0 and $R^T = R^{AG}$ when group means do not re-rank. In the next of this section we will come back on the interpretation of R^B and we will discuss the meaning of R^T showing that neither R^B nor R^T measures re-ranking between income units. Firstly, we derive the formal expressions of R^B and R^T. One obtains R^B by subtracting (16) from (12),

$$R^B = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y{}'\left[\mathbf{S}^B \circ \mathbf{D}_Y\right]\mathbf{p}_Y \tag{21}$$

where $\mathbf{S}^B = {}_{at}\mathbf{S}^B - {}_{bt}\mathbf{S}^B$, while ${}_{at}\mathbf{S}^B = \left(\mathbf{J} - \mathbf{W}_Y\right)\circ \mathbf{A}_Y{}'\mathbf{S}\mathbf{A}_Y$ and ${}_{bt}\mathbf{S}^B = \left(\mathbf{J} - \mathbf{W}_Y\right)\circ \mathbf{E}'\left(\mathbf{A}_X{}'\mathbf{S}\mathbf{A}_X\right)\mathbf{E}$.

One obtains R^T by subtracting (21) from (20),

$$R^T = \frac{1}{2\mu_Y N^2}\mathbf{p}_Y{}'\left[\mathbf{S}^T \circ \mathbf{D}_Y\right]\mathbf{p}_Y \tag{22}$$

where $\mathbf{S}^T = {}_{at}\mathbf{S}^T - {}_{bt}\mathbf{S}^T$, while ${}_{at}\mathbf{S}^T = \left(\mathbf{J} - \mathbf{W}_Y\right)\circ\left(\mathbf{S} - \mathbf{A}_Y{}'\mathbf{S}\mathbf{A}_Y\right)$ and

$$_{bt}\mathbf{S}^T = \left(\mathbf{J} - \mathbf{W}_Y\right) \circ \left[\mathbf{E}'\mathbf{S}\mathbf{E} - \mathbf{E}'\left(\mathbf{A}_X{}'\mathbf{S}\mathbf{A}_X\right)\mathbf{E}\right].$$

Analysing how (21) and (22) yield R^B and R^T requires a detailed discussion of the entries of $\mathbf{S}^T$ and $\mathbf{S}^B$. Here we point out the main observations concerning the interpretation of the entries of $\mathbf{S}^T$ whereas the reader is referred to Appendix 1 for a full discussion on the entries of $\mathbf{S}^T$ and $\mathbf{S}^B$ in terms of re-ranking. The entry (i,j) of $\mathbf{S}^T$, denoted by s_{ij}^T , can be interpreted as follows:[9]

- s_{ij}^T is 0 when both before and after taxes, the relation between the two incomes represents a non-transvariation (see Table 3, Appendix 1, cases *iii-a* and *ii-b*). It is 0 also when the relation between the incomes represents a transvariation both before and after taxes, but neither incomes nor averages re-rank (*i-a*);
- s_{ij}^T is 2 when a *b.t.* transvariation disappears *a.t.*, due to individual income re-ranking (*iv-a*). It is also 2 when a new transvariation is introduced by taxes and there is no transvariation before taxes (*ii-a* and *iii-b*). The new transvariation may derive from individual income re-ranking only (*ii-a*), or from average income re-ranking only (*iii-b*). To distinguish if $s_{ij}^T = 2$ is due to the introduction of a new transvariation or to the elimination of a previous transvariation, we have to look at $_{at}s_{ij}^B$ and $_{bt}s_{ij}^B$.
- s_{ij}^T is –2 when a *b.t.* transvariation disappears due to the re-ranking of group income averages, while involved incomes do not re-rank (*i-b*);
- s_{ij}^T is 4 whenever a transvariation exists both before and after taxes; this result occurs when both incomes and their group averages re-rank (*iv-b*).

A simple numerical example illustrates the interpretation of the entries of $\mathbf{S}^T$. Consider a population partitioned in two groups. The *a.t.* income vector of group 1 is $\mathbf{y}_1' = (5,8,24)$ and that of group 2 is $\mathbf{y}_2' = (4,9,10,32)$. Then, *a.t* it is $\mu_{Y2} > \mu_{Y1}$. Suppose that *b.t.* $\mu_{X2} > \mu_{X1}$, so that group means do not re-rank from *b.t.* to *a.t.* distribution. The vector of ungrouped *a.t.* incomes sorted by increasing order is $\mathbf{y}' = (4,5,8,9,10,24,32)$; the vector of ungrouped *a.t.* incomes sorted by *b.t.* increasing order is $\mathbf{y}_X' = (5,4,10,8,9,24,32)$. Comparing the vectors $\mathbf{y}'$ and $\mathbf{y}_X'$,

[9] The discussion concerns super-diagonal elements of $\mathbf{S}^T$ ($j > i$) only, since $\mathbf{S}^T$ is anti-symmetric.

one immediately notes that across-group re-ranking involves the income pairs (4,5) and (8,10) while within-group re-ranking involves the pair (9,10). The re-ranking between 4 and 5 introduces a transvariation (case *ii-a*); the re-ranking between 8 and 10 eliminates a transvariation (case *iv-a*). In cases *ii-a* and *iv-a* the entries of $\mathbf{S}^T$ equal 2. The matrix $\mathbf{S}^T$ clearly reveals the existence of these rank changes, showing in its super-diagonal entries $s_{1,2}^{AG} = s_{3,5}^{AG} = 2$, while all other entries equal 0. From the definitions of $\mathbf{S}^T$ and $\mathbf{S}^{AG}$ one observes that $\mathbf{S}^T$ coincides with $\mathbf{S}^{AG}$ when means do not re-rank.

$$\mathbf{S}^T = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{S}^{AG}.$$

Suppose now that group means re-rank ($\mu_{Y2} > \mu_{Y1}$ and $\mu_{X2} < \mu_{X1}$). Moreover, suppose that the vector of ungrouped *a.t.* incomes sorted by *b.t.* increasing order becomes $\mathbf{y}'_X = (5,4,10,8,9,32,24)$, while the vector $\mathbf{y}'$ is unchanged. When group means re-rank $\mathbf{S}^T$ differs from $\mathbf{S}^{AG}$. Appling the definitions of $\mathbf{S}^T$ and $\mathbf{S}^{AG}$ one has

$$\mathbf{S}^{AG} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix} \qquad \mathbf{S}^T = \begin{bmatrix} 0 & 4 & 2 & 0 & 0 & 2 & 0 \\ -4 & 0 & 0 & -2 & -2 & 0 & -2 \\ -2 & 0 & 0 & -2 & 0 & 0 & -2 \\ 0 & 2 & 2 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ -2 & 0 & 0 & -2 & -2 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The above discussed cases linked with the elements of $\mathbf{S}^T$ are exemplified in that matrix. For example, the income pair (5,9) falling into case (*i-b*) is associated with $s_{2,4}^T = -2$ (*transvariation disappears but no re-ranking occurs between incomes*); case (*ii-b*) is fulfilled by the pairs (24,32) and (8,10) with $s_{6,7}^T = s_{3,5}^T = 0$ (*no transvariation before and after tax, re-ranking occurs between incomes*); case (*iii-b*) comes true for the pair (4,24) with $s_{1,6}^T = 2$ (*a.t. transvariation, b.t. no* *transvariation, no re-ranking occurs between incomes)*; the pair (4,5) falls into case (*iv-b*) with $s^T_{1,2} = 4$ (*transvariation a.t. and b.t., re-ranking occurs between incomes*).

In the super-diagonal part of $\mathbf{S}^{AG}$ one has $s^{AG}_{1,2} = s^{AG}_{3,5} = s^{AG}_{6,7} = 2$ and 0 in the other entries. The entries different from 0 are associated with the pairs (24,32), (8,10) and (4,5) which represent the three cases of re-ranking between incomes belonging to different groups.

After discussing the formal expressions of R^B and R^T, we face their interpretation. Let us reconsider R^B. We state that R^B evaluates the re-ranking between groups when groups are considered as objects, since it does not evaluate re-ranking between income units. We think that to prove this statement, complicated formal reasoning is not required. A simple example can be much more convincing by showing that one may observe group average income re-ranking even though no re-ranking occurs between any two income receipts belonging to different groups. Suppose there is a population of income receipts partitioned into two groups, each of them containing three income receipts, and let the *b.t.* income vector of group 1 be given by $\mathbf{x}'_1 = (19, 25, 30)$ and that of group 2 by $\mathbf{x}'_1 = (19, 25, 30)$. The *b.t.* average income of group 1 is lower than that of group 2. Assume now that $\mathbf{y}'_1 = (17{,}21{,}24)$ and $\mathbf{y}'_2 = (16{,}19{,}26)$ are the *a.t.* income vectors of groups 1 and 2, respectively. The *a.t.* average income of group 1 is higher than that of group 2: the average incomes re-rank moving from the *b.t.* income distribution to the *a.t.* one. However, no re-ranking occurs both within groups and across groups, therefore implying that the APK re-ranking index equals zero.

This leads us to another important remark. If $R^B > 0$ and R^{AG}=0, the value of R^T must be equal to $-R^B$. It derives that, R^T is not a re-ranking measure in itself. R^T is the quantity we must add to R^B in order to guarantee the identity $R^{AG} = R^B + R^T$. To further prove that R^T is not a re-ranking measure, we analyse the link between re-ranking and transvariation. This occurs because the term R^T is defined as the difference between the measures of the *a.t.* overlapping and the *b.t.* overlapping (evaluated by after-tax incomes), and the overlapping measure defines as twice the weighed sum of transvariations.

When group averages do not re-rank, incomes belonging to different groups re-rank if a *b.t.* transvariation becomes an *a.t.* non-transvariation or, vice versa, when a *b.t.* non-transvariation becomes an *a.t.* transvariation (cases *ii-a* and *iv-a*). There is no re-ranking when either a *b.t.* transvariation remains a transvariation after taxation (case *i-a*) or a *b.t.* non-transvariation still is a non-transvariation after taxation (case *iii-a*). Then, when group averages do not re-rank, a change in the relationship of transvariation between two incomes signals that the two incomes have permuted their reciprocal positions. The entries of $\mathbf{S}^T$ are $s_{ij}^T = 2$ in cases (*ii-a*) and (*iv-a*) while one has =0 in cases (*i-a*) and (*iii-a*) (see the example reported above). $\mathbf{S}^T$ mimics exactly $\mathbf{S}^{AG}$ and R^T equals R^{AG}, that is the re-ranking measure ($R^T \equiv R^{AG}$).

When group averages re-rank, changes in transvariation relationships do not imply any re-ranking. There is re-ranking between income receivers belonging to different groups if taxes neither introduce new transvariations (case *ii-b*) nor eliminate the existing ones (case *iv-b*). Vice-versa, no re-ranking occurs when either *b.t.* transvariations disappear (case *i-b*) or new transvariations are introduced (case *iii-b*). Then, when group averages re-rank, the re-ranking between income receivers belonging to different groups is linked to the persistence of either transvariations or non-transvariations. It derives that all the incomes of the two groups might re-rank together with their group averages, in the absence of any transvariation both before and after taxes. Being G_Y^T the weighted sum of transvariations, $R^T = G_Y^T - C_{Y|X}^T$ cannot be a re-ranking measure. An example may clarify this point. Let us consider a case of group mean re-ranking characterized by absence of *b.t.* overlapping and thus by no *b.t.* transvariations (i.e., $G_X^T = 0,\ C_{Y|X}^T = 0$). Suppose that groups overlap after taxes (i.e., there are *a.t* transvariations) so that one has $R^T = G_Y^T$. Now let us recall that *b.t.* non-transvariations become *a.t* transvariations when group means re-rank. It derives that being $G_X^T = 0$, if no other things happen, all the income pairs belong to the case (*iii-b*), with $s_{ij}^T = 2$. Only if some income pairs re-rank (i.e., from being a transvariation by effect of mean re-ranking it turns to be a non-transvariation by effect of its own re-ranking), we may observe some entries $s_{ij}^T \neq 2$: more precisely, we may observe $s_{ij}^T = 0$ (case *ii-b*). Thus, in $\mathbf{S}^T$ we observe 2 if income units do not re-rank and 0 when income units re-rank. It is then clear that R^T does not measure any form of re-ranking: $R^T = G_Y^T$ measures the amount of *no re-ranking.* The re-ranking between incomes belonging to different groups is evaluated by R^{AG}.[10]

Summing up we can say that neither R^B nor R^T measures re-ranking between

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[10] We disagree with Urban and Lambert (2008). These authors face a case like that discussed in the example and suggest the following re-ranking decomposition (using our notation) $R^{APK} = R^W + R^B + G_Y^T$.

incomes belonging to different groups. The measure of this type of re-ranking is R^{AG}. It follows that R^{APK} can be decomposed into the sum of two re-ranking measures R^{W} and R^{AG}.

4. EMPIRICAL APPLICATION

We applied the re-ranking decomposition to a household sub-sample selected from the 2007 survey of the Panel Study on Income Dynamics (PSID). The survey provides income, employment and demographic data for individuals and households included in a representative sample of the United States population (PSID, 2012). We focused on household pre-government income and post-government income: the former is the combined income before taxes and government transfers of household members aged 15 or older, the latter is the combined income after taxes and government transfers of household members aged 15 or older. From the household panel of size 8,289, we selected households with children. We then excluded households with non-positive incomes. The result was 3,505 households, which were partitioned into three groups by the number of children in a household: group 1 included households with one child; group 2 was comprised of households with two children; group 3 was composed of households with three or more children. In order to account for different household sizes and economies of scale, incomes were adjusted using the OECD square root scale (Schluter and Trede, 2003) which divides household income by the square root of household size. In all calculations we used cross-sectional weights provided by the survey. Table 1 reports descriptive statistics for the three groups. From Table 1, we observe that no re-ranking occurs among group mean incomes, therefore one has $R^{B} = 0$ and $R^{T} = R^{AG}$.

Table 1: Descriptive statistics for pre-government (pre) and post-government (post) equivalised incomes, USA, 2007.

statistic	group 1		group 2		group 3		total	
	pre	post	pre	post	pre	post	pre	post
mean	49,281.64	39,532.33	43,624.23	34,874.02	32,565.64	27,223.84	43,778.11	35,294.64
std. dev.	315,032.69	213,191.43	197,162.54	128,228.55	154,369.64	121,966.14	247,184.15	168,579.73
min	14.14	100	28.86	180	8.16	93.91	8.16	93.91
max	1,677,500	1,023,945.5	671,000	419,679.63	607,339.98	534,413.24	1,677,500	1,023,945
N	1,453		1,272		780		3,505	

Source: calculations on PSID data.

In the following, we analyse the re-ranking effect in the move from the pre-government to the post-government income distribution. To thoroughly discuss this effect we will consider not only the re-ranking measures but we take also into account the number of re-rankings occurred among pairs of income units. This number is defined re-ranking incidence by van de Ven and Creedy (2005). To avoid misunderstandings, the re-ranking measure R^{APK} is referred to as the re-ranking intensity.

We can decompose the re-ranking incidence by slightly modifying the matrix formulae introduced in Section 3 for the APK re-ranking components. The incidence of re-ranking within group *h*, $n\left(R_h^W\right)$, is given by

$$n\left(R_h^W\right) = \frac{1}{4}\mathbf{j}'\left[\mathbf{S}_h^W \circ \mathbf{S}\right]\mathbf{j}, \tag{23}$$

and the incidence of re-ranking across groups *h* and *g* is

$$n\left(R_{hg}^{AG}\right) = \frac{1}{4}\mathbf{j}'\left[\mathbf{S}_{hg}^{AG} \circ \mathbf{S}\right]\mathbf{j}. \tag{24}$$

Equations (23) and (24) calculate within-group and across-group re-ranking incidences in absolute terms. We can express the within-group re-ranking incidence in relative terms dividing $n\left(R_h^W\right)$ by its maximum that is equal to $N_h(N_h-1)/2$,[11]

$$f\left(R_h^W\right) = \frac{1}{2N_h\left(N_h-1\right)}\mathbf{j}'\left[\mathbf{S}_h^W \circ \mathbf{S}\right]\mathbf{j}. \tag{25}$$

The relative incidence of re-ranking occurring across groups *h* and *g* is

$$f\left(R_{hg}^{AG}\right) = \frac{1}{4N_hN_g}\mathbf{j}'\left[\mathbf{S}_{hg}^{AG} \circ \mathbf{S}\right]\mathbf{j}. \tag{26}$$

In (26), N_hN_g is the maximum incidence of re-ranking that occurs when complete re-ranking exists among income receivers of group *h* and those of group *g*. It is worth noting that the relative incidence of re-ranking allows to compare the incidence of re-ranking within and across groups, even when group sizes are different.

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[11] Given the group size N_h, when complete re-ranking occurs among income receivers the incidence of re-ranking within that group is $n\left(R_h^W\right) = \sum_{i=1}^{N_h-1} i = N_h\left(N_h-1\right)/2$.

Table 2 shows the decompositions of Gini inequality indices and those of re-ranking intensities and incidences, referred to pre-government and post-government income distributions. As expected, the post-government Gini index is less than the pre-government one, due to the equalising effect of taxes and transfers. The re-ranking intensity is less than 1% of the post-government Gini index; it is less than 6% of the redistributive effect of taxes and transfers, evaluated by $G_{pre} - G_{post}$. Analysing the re-ranking intensity decomposition, one observes that the 63% is due to the across-group component and only the 37% is due to the within-group component. Decomposing the ratios R^{APK}/G_{post} and $R^{APK}/(G_{pre} - G_{post})$ into their within-group and across-group components, one obtain ratios that are quite similar. Table 2 shows that the range is 0.896-0.900 for the former ratio and it is 5.647-5.830 for the latter. Groups 1 and 3 show ratios R^{APK}/G_{post} equal to 1.067% and 0.996% respectively, and ratios $R^{APK}/(G_{pre} - G_{post})$ equal to 7.132% and 7.747% respectively. These ratios are much greater than those observed for group 2, where (R^{APK}/G_{post})=0.614% and $[R^{APK}/(G_{pre} - G_{post})]$=3.353%.

Column 6 in Table 2 shows that re-ranking in absolute values mainly occurs across groups, especially across groups 1 and 2 (81,244). This is not surprising since the two groups are the most numerous, therefore, the number of possible re-ranking occurrences is the largest (N_1N_2= 1,848,216). When the incidence of re-ranking across groups is concerned, we can split re-ranking occurrences into the sum of the number of transvariations eliminated, $n(T_E)$, and the number of transvariations introduced by taxes and transfers, $n(T_I)$. We note that the re-ranking incidence across groups 1 and 2 is caused by T_E and T_I in equal shares. The re-ranking incidence across groups 1 and 3 is mainly due to transvariations eliminated by taxes and transfers, and the same occurs across groups 2 and 3. In fact, for the group pairs 1-3 and 2-3 the disproportion between $n(T_I)$ and $n(T_E)$ is remarkable: $n(T_I)/n(T_E)$ is 73.9% for the group pair 2-3, and 81% for the group pair 1-3. To verify if these ratios signal that taxes and transfers change the overlapping between groups, we have to consider the re-ranking intensity of T_I and T_E.

Column 8 in Table 2 shows the ratios between the intensity of re-ranking due to the new post-government transvariations, $R^{APK}(T_I)$, and that ascribable to the eliminated pre-government transvariations, $R^{APK}(T_E)$, for the various across-group components. For the across-group component 2-3, $R^{APK}(T_I)/R^{APK}(T_E)$ is close to the value of $n(T_I)/n(T_E)$. Differently from the across-group component 2-3, the re-ranking intensity across groups 1 and 2 due to new post-government transvariations is much lower than that ascribable to the pre-government transvariations eliminated by taxes and transfers, being $R^{APK}(T_I)/ R^{APK}(T_E)$ equal to 64.7%, even though the number of transvariations introduced by taxes and transfers is almost equal to the number of eliminated pre-government trasvariations. The same occurs for the re-ranking across groups 1 and 3, since the re-ranking intensity ascribable to the new post-government transvariations is much less than that attributable to the eliminated pre-government transvariations: being $n(T_I)/n(T_E)$ equal to 81% and $R^{APK}(T_I)/R^{APK}(T_E)$ equal to 53.9%. Given these results it seems possible to say that taxes and transfers change the overlapping among groups when the change is evaluated by post-government incomes. In particular, the overlapping between group 1 and the other two groups decreases. In other words, following Yitzhaki (1994) we can say that taxes and transfers augment stratification among groups.

Table 2: Gini and re-ranking decompositions by group, USA, 2007.

components ↓		(1) G_{pre}	(2) G_{post}	(3) R^{APK}	(4) $\frac{R^{APK}}{G_{post}}$	(5) $\frac{R^{APK}}{G_{pre}-G_{post}}$	(6) $n(R^{APK})$	(7) $n(T_I)/n(T_E)$	(8) $R^{APK}(T_I)/R^{APK}(T_E)$	(9) $f(R^{APK})$
within-group	1	9.830	8.554	0.091	1.067	7.132	46,623	.	.	4.4
	2	6.363	5.378	0.033	0.614	3.353	34,722	.	.	4.3
	3	1.449	1.284	0.013	0.996	7.747	15,868	.	.	5.2
within-group		17.642	15.216	0.137	0.900	5.647	97,213			4.5
across-group	1-2	15.918	13.684	0.122	0.892	5.449	81,244	100.3	64.7	4.4
	1-3	8.018	7.071	0.072	1.017	7.593	51,967	81.0	53.9	5.2
	2-3	6.309	5.459	0.041	0.746	4.789	46,589	73.9	71.2	4.7
across-group		30.245	26.214	0.235	0.896	5.830	179,800	87.2	62.6	4.5
Total		47.886	41.43	0.371	0.897	5.754	277,013			4.5

Source: calculations on PSID data. Indices and ratios are multiplied by 100.

5. CONCLUSIONS

The article has accomplished two tasks. First, it provides a matrix-based approach for the calculation of the Gini and concentration indices. The matrix expressions of the two indices are algebraically decomposable into the sum of three components, each of them expressed as a function of the same matrix. This makes the components of the two indices easily comparable.

Second, the article provides a new decomposition by group for the APK re-ranking measure. As it is known, the APK re-ranking is measured by the difference between the Gini and the concentration index: both indices are decomposable as the sum of three components, consequently the APK re-ranking index too is algebraically decomposable into three parts. However, we show that when the measurement of re-ranking concerns income units gathered into groups, the APK re-ranking can meaningfully be decomposed only into two parts: the measure of re-ranking within groups and the measure of re-ranking across groups. As illustrated by the application in Section 4, our matrix approach proves to be quite powerful: in addition to providing the various re-ranking measures, it also allows to count the number of units involved in re-rankings, that is the re-ranking incidence.

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APPENDIX 1

We consider two groups and a pair of income receivers belonging to different groups. We then analyse the rank changes induced by taxation both with and without group mean re-ranking. Considering the super-diagonal entries of $\mathbf{D}_Y$ related to incomes belonging to different groups, eight different cases can occur. For each of them in Table 3 we give the values in the cells of the matrices $\mathbf{S}^{AG}$, ${}_{at}\mathbf{S}^{B}$, ${}_{bt}\mathbf{S}^{B}$, $\mathbf{S}^{B}$, ${}_{at}\mathbf{S}^{T}$, ${}_{bt}\mathbf{S}^{T}$, $\mathbf{S}^{T}$.

To obtain Table 3 each income unit y_j $(j \equiv r_Y(y_j))$ has to be identified by both the group index and the within group individual index. Then, denoting $y_{h,l} \equiv y_j$ and $y_{g,m} \equiv y_i$, one rewrites the super-diagonal elements of $(d_{ij}^Y = y_j - y_i \geq 0,\ j > i)$, with $r_Y(y_{h,l}) > r_Y(y_{g,m})$ and $h \neq g$.

Table 3: Super diagonal entries of S^{AG}, ${}_{at}S^{B}$, ${}_{bt}S^{B}$, S^{B}, ${}_{at}S^{T}$, ${}_{bt}S^{T}$, S^{T}.

				$j>i : r_Y(y_{h,l}) > r_Y(y_{g,m}),\ y_{h,l} \equiv y_j,\ y_{g,m} \equiv y_i,\ h \neq g$						
				s_{ij}^{AG}	${}_{at}s_{ij}^{B}$	${}_{bt}s_{ij}^{B}$	s_{ij}^{B}	${}_{at}s_{ij}^{T}$	${}_{bt}s_{ij}^{T}$	s_{ij}^{T}
(i) $r_X(x_{h,l})>r_X(x_{g,m})$	*(i-a)*	$\mu_{Xh}<\mu_{Xg}$	$\mu_{Yh}\leq\mu_{Yg}$	0	-1	-1	0	2	2	0
	(i-b)	$\mu_{Xh}<\mu_{Xg}$	$\mu_{Yh}>\mu_{Yg}$	0	1	-1	2	0	2	-2
(ii) $r_X(x_{h,l})<r_X(x_{g,m})$	*(ii-a)*	$\mu_{Xh}<\mu_{Xg}$	$\mu_{Yh}\leq\mu_{Yg}$	2	-1	-1	0	2	0	2
	(ii-b)	$\mu_{Xh}<\mu_{Xg}$	$\mu_{Yh}>\mu_{Yg}$	2	1	-1	2	0	0	0
(iii) $r_X(x_{h,l})>r_X(x_{g,m})$	*(iii-a)*	$\mu_{Xh}>\mu_{Xg}$	$\mu_{Yh}\geq\mu_{Yg}$	0	1	1	0	0	0	0
	(iii-b)	$\mu_{Xh}>\mu_{Xg}$	$\mu_{Yh}<\mu_{Yg}$	0	-1	1	-2	2	0	2
(iv) $r_X(x_{h,l})<r_X(x_{g,m})$	*(iv-a)*	$\mu_{Xh}>\mu_{Xg}$	$\mu_{Yh}\geq\mu_{Yg}$	2	1	1	0	0	-2	2
	(iv-b)	$\mu_{Xh}>\mu_{Xg}$	$\mu_{Yh}<\mu_{Yg}$	2	-1	1	-2	2	-2	4

In the first column of Table 3, the values of s_{ij}^{AG} signal either the presence or the absence of re-ranking between the two incomes. Whenever *a.t.* incomes permute their reciprocal positions with respect to *b.t.* income parade, one has $s_{ij}^{AG} = 1$; if no re-ranking occurs $s_{ij}^{AG} = 0$. In columns 2 and 3, the terms ${}_{at}s_{ij}^{B}$ and ${}_{bt}s_{ij}^{B}$ signal which group average presents the greater value before and after taxes: one observes ${}_{bt}s_{ij}^{B} = 1$ when $\mu_{Xh} > \mu_{Xg}$ and ${}_{bt}s_{ij}^{B} = -1$ if $\mu_{Xh} < \mu_{Xg}$; analogously ${}_{at}s_{ij}^{B} = 1$ when $\mu_{Yh} > \mu_{Yg}$ and ${}_{at}s_{ij}^{B} = -1$ if $\mu_{Yh} < \mu_{Yg}$. In column 4, $s_{ij}^{B} = \pm 2$ signals group average permutation. More precisely $s_{ij}^{B} = -2$ if before taxes $\mu_{Xh} > \mu_{Xg}$ and after taxes $\mu_{Yh} < \mu_{Yg}$, $s_{ij}^{B} = 2$ if $\mu_{Xh} < \mu_{Xg}$ and $\mu_{Yh} > \mu_{Yg}$.

When group averages do not permute their reciprocal ranks one has ${}_{bt}s_{ij}^{T} = {}_{at}s_{ij}^{T}$ and then $s_{ij}^{B} = 0$.[12] In columns 5 and 6 we have ${}_{at}s_{ij}^{T}$ and ${}_{bt}s_{ij}^{T}$ that are the two components of the term s_{ij}^{T} reported in the last column of Table 3. Both ${}_{at}s_{ij}^{T}$ and ${}_{bt}s_{ij}^{T}$ are equal to zero, if before and after taxes, respectively, the relation between the two incomes does not represent a transvariation. One has ${}_{at}s_{ij}^{T} = 0$ if after taxes the relation between the two incomes does not represent a transvariation. One has ${}_{at}s_{ij}^{T} = 2$ when the relation between the two incomes is a transvariation. Because after-tax incomes in the super-diagonal entries of $\mathbf{D}_Y$ are such that $r_Y(y_{h,l} \equiv y_j) > r_Y(y_{g,m} \equiv y_i)$, we have a transvariation if and only if $\mu_{Yh} < \mu_{Yg}$; it follows that ${}_{at}s_{ij}^{T}$ can equal 0 or 2. The term ${}_{bt}s_{ij}^{T}$ can equal: 0, +2, –2. Analogously to ${}_{at}s_{ij}^{T}$, it is ${}_{bt}s_{ij}^{T} = +2$ when $r_X\left(x_{h,l} \equiv x_j\right) > r_X\left(x_{g,m} \equiv x_i\right)$ and $\mu_{Xh} < \mu_{Xg}$. It is ${}_{bt}s_{ij}^{T} =$ –2 when $r_X\left(x_{h,l} \equiv x_j\right) < r_X\left(x_{g,m} \equiv x_i\right)$, $\mu_{Xh} > \mu_{Xg}$, that is when the *b.t.* rankings are opposite to the *a.t.* ones.

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[12] R^B equals R^{APK} in the particular case that all incomes in a group are substituted by their group average, so it cannot be negative (see e.g. Vernizzi 2009).

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