MAXIMUM LIKELIHOOD ESTIMATE OF MARSHALL-OLKIN COPULA PARAMETER: COMPLETE AND CENSORED SAMPLE

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Abstract *The aim of this paper is the derivation of the maximum likelihood estimators of the Marshal-Olkin copula. This copula comes from the Marshall-Olkin Bivariate Exponential (MOBE) distribution, that has been proposed in reliability analysis to study complex systems in which the components are not independent and it is also used in the extreme value theory. We find the likelihood estimators considering the cases of complete and Type-II censored samples. The Marshall-Olkin copula likelihood function is presented in both cases. A simulation study in the particular context of the MOBE shows the properties of the proposed estimators for full or censored data. Finally, we analyse some data sets for illustrative purpose.*

Keywords: *Copula model, Marshall-Olkin exponential distribution, Reliability analysis, Bivariate Weibull distribution.*

1. INTRODUCTION

The Marshall-Olkin distribution (Marshall and Olkin, 1967) is a bivariate exponential distribution usually used in reliability analysis to study complex systems with dependent life time random variables of the components. Let X and Y be the lifetime random variables of two components in the complex system. In according to the Marshall-Olkin distribution, the reliability function is

$$\overline{F}(x,y) \quad = \quad P(X>x,Y>y) = \exp\{-\lambda_1 x - \lambda_2 y - \lambda_3 \max(x,y)\} \qquad (1)$$

with $x \geq 0$, $y \geq 0$, $\lambda_1, \lambda_2 > 0$ and $\lambda_3 \geq 0$. The parameters λ_1 and λ_2 are reliability parameters related to the failures of the first and second components, respectively, while the parameter λ_3 is related to the contemporary failures of both components. If $\lambda_3 = 0$ the marginal random variables are independent and so the failure of one of the components does not affect the failure of the other.

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The marginal random variables are exponential distributed with $\lambda_X^* = \lambda_1 + \lambda_3$ and $\lambda_Y^* = \lambda_2 + \lambda_3$ biing the rates of failure parameters of the two components. The marginal random variables are positively correlated.

An interesting feature of this distribution is that the bivariate variable is not absolutely continuous in $\mathscr{R}^2$. It is absolutely continuous in the region $\{(x,y) : x > y \cup y > x\}$ and it is singular in the region defined by the condition $x = y$. The event $X = Y$ occurs when the failure is caused by a simultaneous shock felt by both components. This event has a positive probability $P(Y = X) = \frac{\lambda_3}{\lambda}$ with $\lambda = \lambda_1 + \lambda_2 + \lambda_3$, in the case $\lambda_3 > 0$. For this reason we can write the reliability function as a linear combination of the absolutely continuous part $\overline{F}_a$ and the singular one $\overline{F}_s$:

$$\overline{F}(x,y) = \frac{\lambda_1 + \lambda_2}{\lambda}\overline{F}_a(x,y) + \frac{\lambda_3}{\lambda}\overline{F}_s(x,y), \tag{2}$$

where $\overline{F}_s(x) = \exp\{-\lambda \max(x,y)\}$ and $\overline{F}_a$ can be obtained by subtraction. Therefore for $x = y$ the distribution is not differentiable with respect to two-dimensional Lebesgue measure.

In bivariate and multivariate distributions the dependence structure existing between the marginal random variables is described by a copula. The copula is a helpful tool for handling multivariate distributions with given univariate marginals (Nelsen, 2006; Fisher, 1997). The use of the copula simplifies the model specification and gives a general class of distributions with the same dependent structure and arbitrary marginal distributions.

In this paper we consider the copula of the Marshall-Olkin Bivariate Exponential (MOBE) distribution. This copula is called Marshall and Olkin Copula (MOC).

In the literature several papers discuss the problem of the estimation of two bivariate distributions whose dependent structure is represented by the MOC: the MOBE distribution and the Marshal-Olkin Bivariate Weibull (MOBWE) distribution. Generally the problem is solved by maximum likelihood method, however the solution does not always exists (Beims *et al.,* 1973) and it cannot be obtained in explicit form. Iterative procedure and EM algorithm can be used (Kundu and Dey, 2009). Many works have been proposed on MOBE distributions but few works have considered for distribution different from MOBE with dependent structure given by MOC, especially for censored data (see, among the others, Chiodini (1998), Osmetti and Chiodini (2008), and Osmetti and Chiodini (2011)).

In literature the *Inference Function of Margins (IFM) likelihood* method (Joe, 1997) is used to estimate multivariate distributions. This method consists of estimating parameters of the marginal distributions from maximizing univariate likelihoods and then estimating dependence parameters by an optimization of the multivariate likelihood (Joe and Xu, 1996). This approach is applied to models in which the univariate margins are separated from the dependent structure, for example when the dependence is summarised by a copula (Kim *et al*, 2007). In order to apply this approach to estimate the parameters of the distributions whose dependent structure is represented by the MOC, we find the MOC parameter estimator.

In this paper we obtain the likelihood estimators of MOC parameters for complete and censored data. Monte Carlo simulations are performed for the MOBE, in order to assess the performance of the IFM procedure by using the proposed copula parameter estimator. Furthermore, the asymptotic properties of the MOC parameters are analysed having observed either complete or censored data. Moreover, our proposal is compared with EM algorithm suggested by Kundu and Dey (2009) for the MOBE.

We highlight that the estimation procedure is applicable to the general case of distributions whose dependent structure is represented by the MOC. Therefore, other simulation studies could be developed for distributions different from MOBE, with the same MOC but different marginal distributions. The simulation results of the MOC estimate obtained for these distributions are similar to the ones obtained for the MOBE. For this reason we present only the simulation results obtained for MOBE.

This paper is organized as follows. In the next section we present the characteristics of the MOC. In Section 3 we present the likelihood function of the copula and we obtain the estimators of the MOC parameter. Moreover, in Section 4 we consider the problem of the estimation of the MOC parameter for censored data. A discussion of the Type-II censoring is also considered.

In Section 5 we present the simulation results. In Section 6 our proposal is applied to empirical data. Finally, present our the conclusions and an appendix.

2. THE MARSHALL-OLKIN COPULA

Every bivariate and multivariate cumulative distribution function F and therefore every reliability function $\overline{F}$ can be treated as the result of two components: the marginal distributions and the dependence structure. The copula describes the way in which the marginals are linked together on the basis of their association to construct the cumulative bivariate distribution functions or bivariate reliability functions (for a mathematically definition see Fisher, 1997, and Nelsen, 2006).

A bivariate copula is a function $C : I^2 \to I$, with $I^2 = [0,1] \times [0,1]$ and $I = [0,1]$, that has all the properties of a cumulative distribution function. In particular it is the cumulative bivariate distribution function of a random variable (U,V) with uniform marginal random variables in [0,1]

$$C(u,v) = P(U \leq u, V \leq v), \quad 0 \leq u \leq 1 \quad 0 \leq v \leq 1$$

To better understand the copula model we recall the Sklar's theorem (Sklar, 1959).

Theorem 2.1 (Sklar) *Let (X,Y) a bivariate random variable with joint distribution function $F_{X,Y}(x,y)$ and marginals $F_X(x)$ and $F_Y(y)$. It exists a copula function $C : I^2 \to I$ such that $\forall x, y \in \mathscr{R}$*

$$F_{X,Y}(x,y) = C(F_X(x), F_Y(y)) \tag{3}$$

If $F_X(x)$ and $F_Y(y)$ are continuous then the copula C is unique. Conversely if C is a copula and $F_X(x)$ and $F_Y(y)$ are marginal distribution functions, then the $F_{X,Y}(x,y)$ in (3) is a joint distribution function.

Moreover, if $F_X(x)$ and $F_Y(y)$ are continuous the copula can be found by the inverse of (3)

$$C(u,v) = F_{X,Y}(F^{-1}(u), F^{-1}(v)) \tag{4}$$

with $u = F_X(x)$ and $v = F_Y(y)$.

In the reliability analysis it is often convenient to express a joint survival function $\overline{F}_{X,Y}(x,y)$ as a copula of its marginal survival functions $\overline{F}_X(x)$ and $\overline{F}_Y(y)$; more specifically a function $\widehat{C} : I^2 \to I$ exists, such that

$$\overline{F}_{X,Y}(x,y) = \widehat{C}(\overline{F}_X(x), \overline{F}_Y(y)) \tag{5}$$

Function $\widehat{C}$ is called the survival copula.
Starting from the MOBE distribution in (1) we obtain the survival Marshall-Olkin copula (MOC). Considering the easy case of two exchangeable marginal random variables, the MOC is

$$C(u,v) = uv \min(u^{-\theta}, v^{-\theta}) \tag{6}$$

The model appeared first in Cuadras and Augé (1981). The parameter θ has values in $[0,1]$ and reflects the dependence structure existing between the marginals, that are *positively dependent*: if $\theta = 0$ the variables are independent and the copula is an independence copula $C(u,v) = \prod = (u,v)$, if $\theta = 1$ the variables are co-monotonic and the copula is $C(u,v) = M(u,v) = min(u,v)$. For different values of θ we find several copulae in the Fréchet-Hoeffding class

$$\prod(u,v) \leq C(u,v) \leq M(u,v).$$

By the copula we can calculate the association measures between the variables: the Spearman correlation coefficient $\rho_s = \frac{3}{(4/\theta)-1} \geq 0$ and the Kendall-τ $\tau = \frac{\theta}{2-\theta} \geq 0$. For $\theta \to 0$, then $\rho_s, \tau \to 0$ and for $\theta \to 1$ then $\rho_s, \tau \to 1$. The marginal random variables are positively associated. The MOC is, moreover, an extreme value copula because the copula is max-stable: hence for any positive real n we have $C(u,v) = C^n(u^{1/n}, v^{1/n})$ $\forall u, v \in I$.

3. THE MAXIMUM LIKELIHOOD ESTIMATOR FOR THE MARSHALL-OLKIN COPULA PARAMETER

In literature the estimation of multivariate distributions or copula parameters is usually performed by the maximum likelihood method (for the copula see Shih and Louis (1995), Xu (1996), Joe (1997) and for the MOBE distribution see Proschan and Sullo (1960), Bhattacharyya and Johnson (1972)). The maximization problem could be difficult to solve when the dimension is high and the number of parameters is large and an iterative procedure is necessary. The use of copula suggests the methods of *inference functions of margins* or IFM method, (see McLeish and Small (1988), Xu (1996) and Joe (1997)). With this approach we can estimate the parameters of multivariate distributions in two steps: in the first step we estimate the marginal distribution parameters and then in the second step we estimate only the copula parameter.

Let $\underline{X} = (X_1, X_2, ..., X_k)$ a multivariate random variable with cumulative distribution function

$$F_{\underline{X}}(\underline{x}, \lambda_1, .., \lambda_k, \theta) = C(F_{X_1}(x_1, \lambda_1), F_{X_2}(x_2, \lambda_2), ..., F_{X_k}(x_k, \lambda_k), \theta) \quad (7)$$

F_{X_j} for $j = 1, 2, ..., k$ are the absolute continuous marginals with density function $f_{X_j}(x_j, \lambda_j)$ dependent on λ_j parameter in parametric space Λ_j. C is a copula with parameter $\theta \in \Theta$ and the density c.

Consider the k log-likelihood functions for the univariate margins

$$l_j(\lambda_j, \underline{x}) = \sum_{i=1}^{n} \ln f_{X_j}(x_{ij}, \lambda_j) \quad (8)$$

with *j=1,2,...,k* and the log-likelihood function for the joint distribution

$$l(\underline{\lambda},\theta,\underline{x}) = \sum_{i=1}^{n} \ln f(x_{i1},...,x_{ik},\lambda_1,...,\lambda_k,\theta) \tag{9}$$

with $f(\underline{x},\lambda_1,...,\lambda_k,\theta) = c(F_{X_1}(x_1,\lambda_1),...,F_{X_k}(x_k,\lambda_k),\theta)\prod_{j=1}^{k} f_{X_j}(x_j,\lambda_j)$.

The IFM method consists of doing k separate optimizations of the univariate likelihoods to get estimates $\hat{\lambda}_1,\hat{\lambda}_2,...,\hat{\lambda}_k$, followed by an optimization of the conditioned log-likelihood

$$l(\theta,\underline{x}|\hat{\underline{\lambda}}) = \sum_{i=1}^{n} \ln c(F_{X_1}(x_{i1},\hat{\lambda}_1),...,F_{X_k}(x_{ik},\hat{\lambda}_k),\theta), \tag{10}$$

to get $\hat{\theta}$. Under regularity conditions, the estimates $(\hat{\lambda}_1,\hat{\lambda}_2,...,\hat{\lambda}_k,\hat{\theta})$ are so the solutions of

$$Step\ I: \ \frac{\partial l_j(\lambda_j,\underline{x})}{\partial \lambda_j} = 0 \ \forall j \quad Step\ II: \ \frac{\partial l(\theta,\underline{x}|\hat{\underline{\lambda}})}{\partial \theta} = 0.$$

This procedure is computationally simpler than the usual maximum likelihood one (where all parameters are estimated simultaneously), reducing the computational difficulty and the waste of time. The IFM estimators are different from the ML estimators but still have good properties. The property of consistency is studied by Xu (1996) and the asymptotical efficiency by Godambe (1960). The asymptotic distribution of these estimators is a normal distribution with variance and covariance matrix equal to the inverse of Godambe's information matrix Detailed expression can be found in Joe (1997). The maximum likelihood estimator is more efficient than the IFM one. However, Joe and Xu (1996) show that the IFM estimator is highly efficient. The comparison is made through Monte Carlo simulations: the relative efficiency, measured by the ratio of mean square errors of the IFM estimator to the MLE is close to 1.

Moreover, compared to the MLE estimator, the IFM estimator performs better in numerical computations.

3.1. LIKELIHOOD FUNCTION OF MARSHALL-OLKIN COPULA: COMPLETE SAMPLING

To construct the log-likelihood function we need the density function of the copula. First we find the density function of the MOBE distribution, and then, following the same procedure, we calculate the density and the log-likelihood functions of the correspondent copula.

We stated that the Marshall-Olkin random variable is not absolutely continuous in R^2 in respect to the Lebesgue measure. It is possible, however, to specify a density function with regard to the dominating measure, defined as follows (Proschan and Sullo, 1960): let μ_2 denote a 2-dimensional Lebesgue measure and B_2^+ the Borel $\sigma-$algebra in R_k^+, then we define a measure μ by

$$\mu(B) = \mu_2(B) + \mu_1\left(B \cap \left\{x : (x,x) \in R_2^+\right\}\right) \tag{11}$$

for each $B \in B_2^+$, where μ_1 is the Lebesgue measure on the real line. We note that the cumulative distribution function of Marshall and Olkin is absolutely continuous in respect to the measure μ.

Hence the density function is given by

$$f(x,y) = \begin{cases} \lambda_1(\lambda_2+\lambda_3)\overline{F}(x,y) & y > x \\ \lambda_2(\lambda_1+\lambda_3)\overline{F}(x,y) & y < x \\ \lambda_3\overline{F}(x,y) & y = x \end{cases} \tag{12}$$

Now we draw a sample of n iid observations from the Marshall-Olkin random variable (X,Y). Let n_1, n_2 and n_3 be the number of observations satisfying, respectively, the statements $x_i < y_i$, $x_i > y_i$ e $x_i = y_i$ for $i = 1,2,..,n$, such that $n = n_1 + n_2 + n_3$.

The likelihood function $L(\lambda_1,\lambda_2,\lambda_3,\theta,\underline{x},\underline{y}) = \prod_{i=1}^{n} f(x_i,y_i)$ is given by

$$\begin{aligned} L(\lambda_1,\lambda_2,\lambda_3,\underline{x},\underline{y}) &= [\lambda_1(\lambda_2+\lambda_3)]^{n_1}[\lambda_2(\lambda_1+\lambda_3)]^{n_2}(\lambda_3)^{n_3} \\ &\quad \exp\left\{-\lambda_1\sum_{i=1}^{n}x_i - \lambda_2\sum_{i=1}^{n}y_i - \lambda_3\sum_{i=1}^{n}\max(x_i,y_i)\right\} \\ &= [\lambda_1(\lambda_2+\lambda_3)]^{n_1}[\lambda_2(\lambda_1+\lambda_3)]^{n_2}(\lambda_3)^{n_3}\prod_{i=1}^{n}\overline{F}(x_i,y_i) \end{aligned}$$

with $\lambda_1,\lambda_2,\lambda_3 > 0$. If n_1, n_2 and n_3 are all non zero, the solution of the maximum likelihood problem is unique and it is the unique root of the likelihood system of equations. If $n_3 = 0$ the solution is $\hat{\lambda}_3 = 0$ and $\hat{\lambda}_1 = \left[\sum_{i=1}^{n} x_i/n\right]^{-1}$ and $\hat{\lambda}_2 = \left[\sum_{i=1}^{n} y_i/n\right]^{-1}$. The problem of the estimation of this model has been studied by Proschan and Sullo (1960), Bhattacharyya and Johnson (1972), Beimes *et al.* (1973).

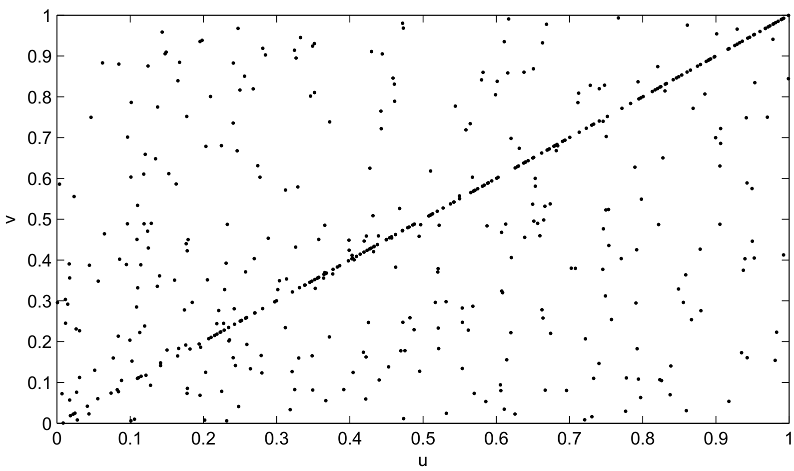

Figure 1: Observations from the Marshall-Olkin copula.

Following the same procedure, we specify the density function of the MOC. We stated that also the copula in (6) has an absolutely continuous part and a singularity for $u = v$ with positive probability. The copula can be so expressed as a liner combination of the absolutely continuous components C_a and the singular one C_s in this way

$$C(u,v) = \frac{2-2\theta}{2-\theta} C_a(u,v) + \frac{\theta}{2-\theta} C_s(u,v) \tag{13}$$

with $C_s(u,v) = [\min(u^\theta, v^\theta)]^{\frac{2-\theta}{\theta}}$. Calculating the derivative of the function (13), we can find the density function of the copula in respect to a dominating measure

$$c_\theta(u,v) = \begin{cases} (1-\theta)u^{-\theta} & u > v \\ (1-\theta)v^{-\theta} & u < v \\ \theta\, u^{1-\theta} & u = v \end{cases} = \begin{cases} (1-\theta)\frac{1}{uv}C_\theta(u,v) & u > v \\ (1-\theta)\frac{1}{uv}C_\theta(u,v) & u < v \\ \theta\frac{1}{u}\, C_\theta(u,v) & u = v \end{cases} \tag{14}$$

with $0 \le u,v \le 1$ and $0 < \theta < 1$.

The copula density function can be constructed also by the Marshall and Olkin density function in (12) (see Nelsen, 2006 and Autin *et al.,* 2010):

$$c(u,v) = \frac{f(F_X^{-1}(u), F_Y^{-1}(v))}{f_X(F_X^{-1}(u)) f_Y(F_Y^{-1}(v))},$$

setting $\lambda_1 = \lambda_2$ and $\theta = \frac{\lambda_3}{\lambda_1+\lambda_3}$, where $f_X(\cdot)$ and $f_Y(\cdot)$ are the density functions of the MOBE marginal exponential distributions.

Now we draw a sample of n iid observations from the random variable (X,Y) generated by the MOC and marginal distributions dependent on the vectors of parameters $\underline{\lambda}_X$ and $\underline{\lambda}_Y$, respectively. Using now the IFM, we estimate in the first step the vectors $\underline{\lambda}_X$ and $\underline{\lambda}_Y$ and in the second step the copula parameter θ. Therefore, let $\hat{u}_i = \bar{F}_X(x_i, \hat{\underline{\lambda}_X})$ and $\hat{v}_i = \bar{F}_Y(y_i, \hat{\underline{\lambda}_Y})$ we maximize the likelihood function with respect to $\theta \in (0,1)$

$$L(\theta|\underline{\hat{u}}, \underline{\hat{v}}) \propto (1-\theta)^{n_1+n_2}\theta^{n_3}\prod_{i=1}^{n} C_\theta(\hat{u}_i, \hat{v}_i). \tag{15}$$

By the logit transformation of θ, $\psi = \ln\left(\frac{\theta}{1-\theta}\right), -\infty < \psi < +\infty$, or $\theta = (1+\exp(-\psi))^{-1}$, we solve the maximization problem with regard to the new parameter $\psi \in \Psi$,

$$\max_{\psi\in\Psi} l(\psi|\hat{u}, \hat{v}). \tag{16}$$

Since the likelihood (15) is conditional on $\underline{\hat{u}}$ and $\underline{\hat{v}}$, then it follows as a consequence that l is the conditional log-likelihood function

$$\begin{aligned} l(\psi|\underline{\hat{u}}, \underline{\hat{v}}) &= k + (n_1+n_2)\ln(1-(1+\exp(-\psi))^{-1}) \\ &+ n_3 \ln((1+\exp(-\psi))^{-1}) + \sum_{i=1}^{n} \ln\left[C_\psi(\hat{u}_i, \hat{v}_i)\right], \end{aligned} \tag{17}$$

where k is a constant. By solving the equation $\frac{\vartheta l(\psi|\hat{u},\hat{v})}{\vartheta \psi} = 0$ we obtain the ψ estimator:

$$\hat{\psi} = -\ln\left[\frac{n - 2n_3 - S_{\min} + \sqrt{n^2 + S_{\min}^2 - S_{\min}(2n-4n_3)}}{2n_3}\right] \tag{18}$$

with $n_3 > 0$, where $S_{\min} = \sum_{i=1}^{n} \min(-\ln(u_i), -\ln(v_i))$. This solution is the unique accepted solution of the maximization problem in the parametric space Ψ (see Appendix 7.1). Thus the maximum likelihood estimator of θ is given by $\hat{\theta} = (1+\exp(-\hat{\psi}))^{-1}$.

4. THE CENSORED SAMPLING

Consider now a case of censored sampling. Supposed we are interested in evaluating the estimates of the parameters $\underline{\lambda}_Y$, $\underline{\lambda}_Y$ and θ using type-II censored data from the

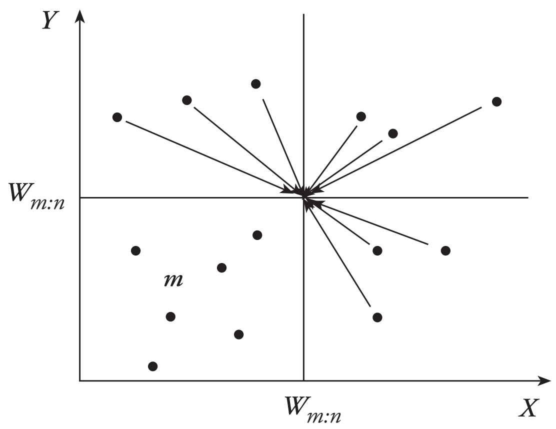

Figure 2: A possible use of sample censored data

random variable (X,Y) generated by the MOC. We stop the experiment once we observe m system failures or a m/n fraction of the system failures. For a univariate random variable X, the censoring consists in stopping the experiment at the m-th observation of the order sample $\{x_{1:n}, x_{2:n}, \ldots, x_{m:n}, \ldots, x_{n:n}\}$. The last observed value $x_{m:n}$ is the length time of the experiment and it is assigned to all the $(n-m)$ unobserved units.

We generalize now the type-II censoring procedure in the bivariate case, stopping the experiment after the observation of m failures of both components. In this case we can treat the sample units in two ways (Chiodini, 1998).

The first situation. Let $\{(x_i, y_i)\ i = 1,\ 2, \ldots, n\}$ be a sample of size n from the bivariate random variable (X,Y). We define a new variable $W = \max(X,Y)$, with sample values $\{w_i = \max(x_i, y_i)\ i = 1, 2, \ldots, n\}$. We order the observations $\{(x_i, y_i)\ i = 1, 2, \ldots, n\}$ according to the order of observations $\{w_{1:n}, w_{2:n}, \ldots, w_{m:n}, \ldots, w_{n:n}\}$, obtaining the order statistics $\{(x_{i:n}, y_{i:n})\ i = 1, 2, \ldots n\}$. We stop the experiment at the point $(x_{m:n}, y_{m:n})$ related to the m-th order value $w_{m:n}$. This value is the length of the experiment and it is assigned to all the $(n-m)$ unobserved units. In this case we use the m observations and the $(n-m)$ unobserved points in both component; the $(n-m)$ unobserved points are set equal to the last observed value $w_{m:n}$. Fig. 2 shows a possible use of the data.

The second situation. We stop the experiment once we observe m failures of both components and then we proceed as shown in Fig. 3. We consider the m values observed in both components, the r values observed for X only (such that $x_i \leq w_{m:n}$ and $y_i > w_{m:n}$), the s values observed for Y only (such that $y_i \leq w_{m:n}$

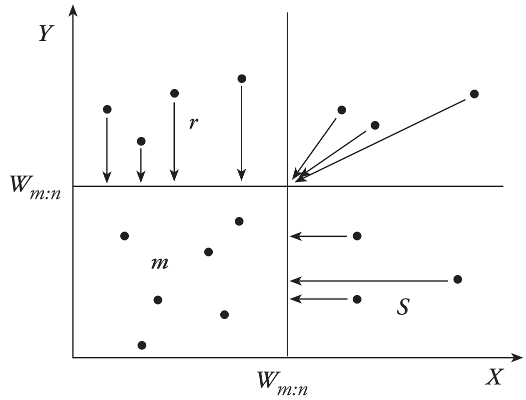

Figure 3: Other use of censored data

and $x_i > w_{m:n}$) and the others unobserved values, in one or both components, that are set equal to $w_{m:n}$. We note in this case that the fraction of the number of the observations, that would be used in the estimation procedure for the marginal random variables, is greater than the fixed fraction m/n (represented in Fig. 2); it is equal to $(m+r)/n$ for X and to $(m+s)/n$ for Y.

4.1. THE LIKELIHOOD FUNCTION OF THE MARSHALL-OLKIN COPULA: CENSORED SAMPLING

Let us assume that the observations of (X,Y) are censored at the value $(w_{m:n}, w_{m:n})$, as shown in Fig. 3. Let $(\Delta^X, \Delta^Y) = (I_{\{X \le w_{m:n}\}}(x), I_{\{Y \le w_{m:n}\}}(y))$, $\bar{\Delta}^X = 1-\Delta^X$ and $\bar{\Delta}^Y = 1-\Delta^Y$, where $I_A(\cdot)$ is the indicator function of the set A. The log-likelihood function of the marginal variable X is

$$\begin{aligned} l(\underline{\lambda}_X, \underline{x}) &= \sum_{i=1}^{n} \ln[(f_x(x_i, \underline{\lambda}_X))^{\Delta_i^X}] + \sum_{i=1}^{n} \ln[(\bar{F}_X(w_{m:n}))^{\bar{\Delta}_i^X}] \\ &= \sum_{i=1}^{m+r} \ln[f_x(x_i, \underline{\lambda}_X)] + (n-m-r)\ln[\bar{F}_X(w_{m:n})]\,. \end{aligned} \tag{19}$$

If the marginal survival distributions $\overline{F}_X(x)$ and $\overline{F}_Y(y)$ are known, the log-likelihood function of bivariate distribution is (Owzar and Sen, 2003)

$$l(\theta, \overline{F}_X, \overline{F}_Y) = \sum_{i=1}^{n} \ln[c(\overline{F}_X(x_i), \overline{F}_Y(y_i))]^{\Delta_i^X \Delta_i^Y} + \sum_{i=1}^{n} \ln[C_1(\overline{F}_X(x_i), \overline{F}_Y(y_i))]^{\overline{\Delta}_i^X \Delta_i^Y} +$$

$$+ \quad \sum_{i=1}^{n} \ln[C_2(\overline{F}_X(x_i), \overline{F}_Y(y_i))]^{\overline{\Delta}_i^Y \Delta_i^X} + \sum_{i=1}^{n} \ln[C(\overline{F}_X(x_i), \overline{F}_Y(y_i))]^{\overline{\Delta}_i^X \overline{\Delta}_i^Y},$$

where $C_1(u,v) = \frac{\partial C(u,v)}{\partial v}$ and $C_2(u,v) = \frac{\partial C(u,v)}{\partial u}$.

We suppose to consider the MOC in (6). Using the IFM method, we estimate in the first step the marginal distribution parameters $\underline{\lambda}_X$ and $\underline{\lambda}_Y$, by maximizing the log-likelihood (19) for X and the analogous log-likelihood $l(\underline{\lambda}_Y, \underline{y})$ for Y. In the second step we estimate the copula parameter θ. Let $\hat{u}_i = \bar{F}_X(x_i, \hat{\underline{\lambda}_X})$, $\hat{v}_i = \bar{F}_Y(y_i, \hat{\underline{\lambda}_Y})$ and let $\theta = (1+\exp(-\psi))^{-1}$, we can estimate the new parameter ψ by the maximization of the log-likelihood function with respect to $\psi \in \Psi$:

$$\begin{aligned} l(\psi|\hat{u},\hat{v}) &= k + (m_1 + m_2 + r + s)\ln[1 - (1+\exp(-\psi))^{-1}] + \\ &+ m_3 \ln[(1+\exp(-\psi))^{-1}] + \sum_{i=1}^{n} \ln(C_\psi(\hat{u}_i, \hat{v}_i)), \end{aligned} \tag{20}$$

where k is a constant and m_1, m_2 and m_3 are the numbers of the m observations (x_i, y_i) (with $x_i \leq w_{m:n}$ and $y_i \leq w_{m:n}$) such that, respectively, $x_i < y_i$, $x_i > y_i$ and $x_i = y_i$.

By solving the equation $\frac{\vartheta l(\psi|\hat{u},\hat{v})}{\vartheta \psi} = 0$ we obtain the estimator $\hat{\psi}$

$$\hat{\psi}_c = -\ln\left[\frac{m+r+s-2m_3-S_{\min}+\sqrt{(m+r+S_{\min}-2m_3)^2+4m_3(m+r+s-m_3)}}{2m_3}\right] \tag{21}$$

with $m_3 > 0$, where

$$S_{\min} = \sum_{i=1}^{m} \min(-\ln(u_i), -\ln(v_i)) + \sum_{i=1}^{r}[-\ln(u_i)] + \sum_{i=1}^{s}[-\ln(v_i)] + (n-m-r-s)w_{m:n}.$$

This is the unique accepted solution of the maximization problem (see Appendix 7.1). The maximum likelihood estimator of θ for censored data is given by $\hat{\theta}_c = (1+\exp(-\hat{\psi}_c))^{-1}$.

Finally, we assume using the sample units as shown in the Fig. 2. By (19) the log-likelihood function of the marginal variable X is

$$\begin{aligned} l(\underline{\lambda}_X, \underline{x}) &= \sum_{i=1}^{n} \ln[(f_x(x_i, \underline{\lambda}_X))^{\Delta_i^X}] + \sum_{i=1}^{n} \ln[(\bar{F}_X(x_i, \underline{\lambda}_X))^{\bar{\Delta}_i^X}] \\ &= \sum_{i=1}^{m} \ln[f_x(x_i, \underline{\lambda}_X)] + (n-m)\ln[\bar{F}_X(w_{m:n}, \underline{\lambda}_X)] \end{aligned} \tag{22}$$

and by (20) the log-likelihood function of the copula is

$$l(\theta|u,v) = \sum_{i=1}^{m} \ln[c(u_i,v_i)] + (n-m)\ln[C(w_{n:m},w_{n:m})], \tag{23}$$

considering the MOC in (6). Using now the IFM method, we estimate firstly the marginal distribution parameter $\underline{\lambda}_X$ and $\underline{\lambda}_Y$ by the maximization of (22) and the analogous log-likelihood for Y, and then the copula parameter θ. Let $\hat{u}_i = \bar{F}_X(x_i,\hat{\underline{\lambda}})$, $\hat{v}_i = \bar{F}_Y(y_i,\hat{\underline{\lambda}})$ and $\theta = (1+\exp(-\psi))^{-1}$, we estimate the new parameter ψ by the maximization of the log-likelihood function with respect to $\psi \in \Psi$:

$$\begin{aligned} l(\psi|\hat{u},\hat{v}) \quad &= \quad k + (m_1+m_2)\ln[1-(1+e^{-\psi})^{-1}] + m_3\ln[(1+e^{-\psi})^{-1}] + \\ &+ \quad \sum_{i=1}^{n} ln\left[C_\psi(\hat{u}_i,\hat{v}_i)\right]. \end{aligned} \tag{24}$$

The unique solution for the maximization problem is

$$\hat{\psi}_c = -\ln\left[\frac{m-2m_3-S_{\min}+\sqrt{(m-S_{\min}-2m_3)^2+4m_3(m-m_3)}}{2m_3}\right] \tag{25}$$

with $S_{\min} = \sum_{i=1}^{m} \min(-\ln(\hat{u}_i), -\ln(\hat{v}_i)) + (n-m)w_{m:n}$ (see Appendix 7.1). The maximum likelihood estimator for censored data is $\hat{\theta}_c = (1+\exp(-\hat{\psi}_c))^{-1}$.

5. SIMULATION

The estimation procedure described above is verified by several simulations (Monte Carlo method), generating 2000 samples with a growing sample size n, from 100 to 1000 from a MOBE distribution with exchangeable marginal variables. The goodness of the obtained estimates is valuated calculating the bias (B) and the mean square error (MSE).

The estimation procedure described in Sections 3 and 4 is also applicable to distributions different from the MOBE (with the same copula and with marginal distributions different from exponential distributions). Therefore, other simulation studies could be enveloped for these distributions (for example for MOBWE). Since all these distributions have the same copula parameter, we obtain, obviously, similar results for the MSE and the bias of the copula parameter estimate. For this reason we present only the simulation results obtained for MOBE.

We consider the case of complete and censored sample (Type-II censoring) generated by a MOBE distribution. In the censored sample 80% of system failures have been observed (see Fig. 3).

The model is generated by the MOC with different parameter values $\theta = 0.1; 0.7; 0.9$ and exponential marginal variables with several values of $\lambda_X^* = \lambda_Y^* = 0.7; 1.3; 2$. The values of θ correspond to the situation of low, mean and high positive association.

Computational aspects

1 Generate data from the MOC by the following algorithm (Devroye, 1987).

- Generate three independent uniform in $(0,1)$ variables r, s ad t.
- Set $z_1 = \min\left(\frac{-ln(r)}{(1-\theta)}, \frac{-ln(t)}{(1-\theta)}\right)$ and $z_2 = \min\left(\frac{-ln(s)}{(1-\theta)}, \frac{-ln(t)}{(1-\theta)}\right)$.
- Set $u = \exp(-z_1)$ and $v = \exp(-z_2)$
- The values of the copula variables are (u, v).

2 Step I: estimate the parameter of the marginal distributions and calculate $\hat{u}_i = \bar{F}_X(x_i, \hat{\underline{\lambda}}_1)$ and $\hat{v}_i = \bar{F}_Y(y_i, \hat{\underline{\lambda}}_2)$

3 Step II: estimate the copula parameter.

4 Repeat 2000 times the procedure and calculate the B and the MSE.

In Appendix 7.2 we describe the Matlab code for the estimation of the copula parameter having observed complete and censored data.

In Tab. 1 we show the results obtained for θ and λ_X^* in the case of complete sample using the two stage maximum likelihood method (IFM method), described in Section 3. We see a good stability of the estimates. Once n increases we see an improvement in the parameter estimates due to a steep decrease of the bias and the MSE, which are inversely proportional to n. Same results are obtained in Tab. 2 for the reliability parameters of MOBE distributions.

Table 1: Maximum likelihood estimation: complete sampling

			λ_X^*		θ		
θ	λ_X^*	n	B	MSE	B	MSE	Time
0.9	0.7	100	-0.0064	0.0050	0.0018	0.0005	7.9222
		500	-0.0021	0.0010	0.0006	0.0001	10.272
		1000	-0.0010	0.0005	0.0002	0.0001	11.303
	1.3	100	-0.0103	0.0171	0.0015	0.0005	4.2758
		500	-0.0021	0.0035	0.0002	0.0001	5.5639
		1000	-0.0011	0.0017	0.0002	0.0001	6.0389
	2.0	100	-0.0182	0.0405	0.0012	0.0006	2.7671
		500	-0.0037	0.0082	0.0003	0.0001	3.5909
		1000	-0.0030	0.0040	0.0003	0.0001	3.9701
0.7	0.7	100	-0.0058	0.0051	0.0047	0.0018	8.2415
		500	-0.0024	0.0009	0.0017	0.0004	10.586
		1000	-0.0013	0.0005	0.0013	0.0002	11.619
	1.3	100	-0.0151	0.0176	0.0033	0.0017	4.4203
		500	-0.0018	0.0034	0.0017	0.0004	5.7073
		1000	-0.0023	0.0018	0.0011	0.0002	6.2695
	2.0	100	-0.0195	0.0408	0.0014	0.0017	2.8797
		500	-0.0033	0.0080	0.0017	0.0004	3.6914
		1000	-0.0008	0.0043	0.0010	0.0002	4.0629
0.1	0.7	100	-0.0084	0.0050	0.0006	0.0016	8.3777
		500	-0.0001	0.0010	0.0003	0.0003	10.703
		1000	-0.0013	0.0005	-0.0001	0.0002	11.665
	1.3	100	-0.0128	0.0168	0.0013	0.0016	4.5478
		500	0.0005	0.0032	0.0003	0.0003	5.7704
		1000	-0.0007	0.0017	0.0003	0.0002	6.3067
	2.0	100	-0.0231	0.0420	0.0016	0.0015	2.9564
		500	-0.0039	0.0077	0.0001	0.0003	3.7378
		1000	-0.0029	0.0040	0.0000	0.0002	4.0892

Table 2: Estimates of reliability parameters: complete sampling

θ	λ_X^*	λ_1	λ_3	n	B	MSE	B	MSE
0.9	0.7	0.07	0.63	100	-0.0016	0.0003	-0.0048	0.0047
				500	-0.0005	0.0001	-0.0015	0.0009
				1000	-0.0002	0.0000	-0.0008	0.0005
	1.3	0.13	1.17	100	-0.0024	0.0010	-0.0079	0.0161
				500	-0.0004	0.0002	-0.0017	0.0033
				1000	-0.0003	0.0001	-0.0008	0.0015
	2.0							
0.2	1.8	100	-0.00340.0025		-0.0148	0.0382		
				500	-0.0007	0.0005	-0.0029	0.0077
				1000	-0.0008	0.0002	-0.0022	0.0038
0.7	0.7	0.21	0.49	100	-0.0044	0.0011	-0.0014	0.0040
				500	-0.0018	0.0002	-0.0006	0.0008
				1000	-0.0012	0.0001	-0.0001	0.0004
	1.3	0.39	0.91	100	-0.0077	0.0036	-0.0075	0.0137
				500	-0.0025	0.0007	0.0007	0.0028
				1000	-0.0019	0.0004	-0.0003	0.0014
	2.0	0.6	1.4	100	-0.0068	0.0083	-0.0128	0.0321
				500	-0.0040	0.0017	0.0007	0.0063
				1000	-0.0021	0.0008	0.0012	0.0034
0.1	0.7	0.63	0.07	100	-0.0075	0.0045	-0.0009	0.0009
				500	-0.0002	0.0009	0.0001	0.0002
				1000	-0.0011	0.0004	-0.0002	0.0001
	1.3	1.17	0.13	100	-0.0127	0.0152	-0.0001	0.0030
				500	0.0001	0.0029	0.0003	0.0006
				1000	-0.0009	0.0015	0.0002	0.0003
	2.0	1.8	0.2	100	-0.0233	0.0380	0.0002	0.0070
				500	-0.0035	0.0071	-0.0003	0.0013
				1000	-0.0025	0.0036	-0.0004	0.0007

We also obtained satisfying results for type-II censored sampling (for the parameters θ and λ_X^* in Tab. 3 and for λ_1 and λ_3 in Tab. 4). In this case we observe a standard increase of the bias and the MSE as compared with complete sampling.

However, we obtained a considerable reduction of the length of the experiment and, therefore, a decrease of the cost of the experiment.

Now we compare the obtained results with the estimators proposed in Section 3 with the ones obtained by EM algorithm of Kundu and Dey (2008) for the

Table 3: Maximum likelihood estimation: censored sampling

θ	λ_X^*	n	B	MSE	B	MSE	Time
0.9	0.7	100	-0.0076	0.0064	0.0009	0.0007	2.4823
		500	-0.0016	0.0012	0.0007	0.0001	2.5092
		1000	-0.0015	0.0006	0.0001	0.0001	2.5063
	1.3	100	-0.0184	0.0208	0.0014	0.0006	1.3327
		500	-0.0048	0.0042	0.0006	0.0001	1.3490
		1000	-0.0014	0.0021	0.0002	0.0001	1.3511
	2.0	100	-0.0312	0.0514	0.0011	0.0007	0.8663
		500	-0.0090	0.0103	0.0005	0.0001	0.8755
		1000	-0.0034	0.0047	0.0003	0.0001	0.8772
0.7	0.7	100	-0.0055	0.0059	0.0050	0.0019	2.8018
		500	-0.0021	0.0011	0.0023	0.0004	2.8204
		1000	-0.0011	0.0006	0.0013	0.0002	2.8216
	1.3	100	-0.0139	0.0204	0.0039	0.0019	1.5037
		500	-0.0025	0.0040	0.0020	0.0004	1.5192
		1000	-0.0024	0.0020	0.0011	0.0002	1.5203
	2.0	100	-0.0211	0.0499	0.0027	0.0019	0.9776
		500	-0.0066	0.0178	0.0020	0.0005	0.9875
		1000	-0.0015	0.0051	0.0015	0.0002	0.9896
0.1	0.7	100	-0.0089	0.0056	0.0007	0.0016	3.1556
		500	-0.0003	0.0011	0.0004	0.0003	3.1830
		1000	-0.0015	0.0005	0.0000	0.0002	3.1836
	1.3	100	-0.0160	0.0186	0.0008	0.0016	1.6932
		500	0.0001	0.0038	0.0004	0.0003	1.7171
		1000	-0.0007	0.0019	0.0001	0.0002	1.7177
	2.0	100	-0.0162	0.0459	0.0008	0.0016	1.1027
		500	-0.0031	0.0087	0.0005	0.0003	1.1140
		1000	-0.0020	0.0046	0.0003	0.0002	1.1144

estimation of the parameters of a MOBE distribution with $\lambda_1 = \lambda_2 = \lambda_3 = 1$. This situation corresponds to the situation $\theta = 0.5$ and $\lambda_X^* = \lambda_Y^* = 2$. The results are obtained with 1000 replications. Tab. 5 shows the values of the average estimates (AM), the MSE and the number of iterations of the EM algorithm (AI). The results obtained with IFM consider obviously only one iteration. In Tab. 5 we note that the results obtained in two steps with the maximum likelihood method (IFM) are appreciable like the ones obtained in more steps with EM methods: the AM of IFM method is close to the real value and the MSE are lower than the ones obtained with EM algorithm. It should be interesting to evaluate the use of

Table 4: Estimates of the reliability parameters: censored sampling

					λ_1		λ_3	
θ	λ_X^*	λ_1	λ_3	n	B	MSE	B	MSE
0.9	0.7	0.07	0.63	100	-0.0010	0.0003	-0.0066	0.0061
				500	-0.0006	0.0001	-0.0010	0.0011
				1000	-0.0002	0.0000	-0.0014	0.0006
	1.3	0.13	1.17	100	-0.0030	0.0011	-0.0154	0.0196
				500	-0.0011	0.0002	-0.0037	0.0040
				1000	-0.0003	0.0001	-0.0010	0.0019
	2.0	0.20	1.80	100	-0.0041	0.0029	-0.0271	0.0487
				500	-0.0016	0.0005	-0.0074	0.0096
				1000	-0.0008	0.0003	-0.0026	0.0044
0.7	0.7	0.21	0.49	100	-0.0044	0.0012	-0.0011	0.0045
				500	-0.0021	0.0002	0.0001	0.0009
				1000	-0.0012	0.0001	0.0001	0.0004
	1.3	0.39	0.91	100	-0.0079	0.0040	-0.0060	0.0158
				500	-0.0031	0.0007	0.0006	0.0031
				1000	-0.0020	0.0004	-0.0004	0.0016
	2.0	0.60	1.40	100	-0.0096	0.0094	-0.0116	0.0382
				500	-0.0067	0.0098	0.0001	0.0071
				1000	-0.0031	0.0009	0.0016	0.0040
0.1	0.7	0.63	0.07	100	-0.0082	0.0049	-0.0007	0.0009
				500	-0.0004	0.0009	0.0001	0.0002
				1000	-0.0013	0.0005	-0.0002	0.0001
	1.3	1.17	0.13	100	-0.0147	0.0163	-0.0012	0.0032
				500	-0.0002	0.0033	0.0004	0.0006
				1000	-0.0007	0.0016	-0.0001	0.0003
	2.0	1.80	0.20	100	-0.0150	0.0392	-0.0012	0.0077
				500	-0.0036	0.0077	0.0004	0.0014
				1000	-0.0023	0.0040	0.0003	0.0007

the estimators obtained with IFM methods like initial guesses in the EM iterative algorithm and check if these new initial values improve the estimates in the EM procedure. Moreover, the EM method proposed by Kundu and Dey (2008) is used for exponential and Weibull bivariate distributions in which the marginal random variable are not necessarily exchangeable. The proposed estimators for the copula, either for complete sample or censored sample, can instead be used for several bivariate distributions, not only for exponential and Weibull, but for the distributions whose dependent structure is represented by the MOC. The copula in (6) is however symmetric. Therefore, it should be interesting to find the IFM estimators for the parameters of distributions generated by an asymmetric Marshall-Olkin copula. This copula depends from two parameters and the region of the discontinuity is not a line but a curve.

Table 5: Comparison between the proposed estimators (IFM) and the estimators of EM algorithm by Kundu and Karlis (KU)

		IFM			EM algorithm			
n		λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	AI
25	AE	1.0288	1.0564	1.0608	1.0335	1.0393	1.0570	20.79
	MSE	(0.1099)	(0.0816)	(0.0866)	(0.1115)	(0.1251)	(0.1055)	
50	AE	1.0147	1.0222	1.0247	1.0291	1.0254	1.0304	19.36
	MSE	(0.0490)	(0.0335)	(0.0349)	(0.0524)	(0.0567)	(0.0475)	
100	AE	1.0082	1.0194	1.0131	1.0129	1.0169	1.0098	18.36
	MSE	(0.0249)	(0.0169)	(0.0166)	(0.0248)	(0.0245)	(0.0224)	
500	AE	1.0005	1.0040	1.0041	1.0046	1.0016	1.0051	17.02
	MSE	(0.0049)	(0.0032)	(0.0036)	(0.0052)	(0.0051)	(0.0045)	

6. APPLICATIONS

In literature there are several distributions with a MOC copula. The Marshall Olkin exponential distribution is typically employed in reliability analysis to study complex systems with dependent components. The bivariate Weibull distributions (MOBWE), obtained by the MOC copula in (6) with marginal Weibull distributions

$$F_X(x) = 1 - \exp\left[-\left(\frac{x}{a_1}\right)^{b_1}\right] \quad F_Y(y) = 1 - \exp\left[-\left(\frac{y}{a_2}\right)^{b_2}\right] \tag{26}$$

is used when the failure rates of the components are not constant and the failures describe a non homogeneous Poisson processes rate.

Real applications described in the literature suggest that these distributions may also be used in providing probabilistic structure to certain sports and medical data (see for example Csörgö and Welsh (1985), Meintanis (2007), Kundu and Dey (2009)).

As our first we use the data set shown in Tab. 6. It represents the 20 observations related with the time taken to call 2 elevators in a hotel's reception, that is the time in minutes between two consecutive calls. X represents the time of the calls concerning the first elevator and Y the time for the second elevator. We have 20 observations. All events are possible $x > y$, $x < y$ and $x = y$. We suppose that the observations have a Marshall-Olkin bivariate distribution generated by a MOC and exponential marginal random variables. The estimates of the marginal distribution parameters λ_X^*, λ_Y^* and of θ, obtained with the described procedure, are $\widehat{\lambda}_X^* = \widehat{\lambda}_Y^* = 0.16$ and $\widehat{\theta} = 0.52$, respectively. The value of θ describe a mean grade

Table 6: Observed data

Call	X	Y	Call	X	Y
1	8	7	11	1	3
2	10	10	12	11	12
3	10	12	13	12	12
4	5	4	14	6	5
5	6	6	15	8	8
6	1	2	16	11	6
7	2	4	17	1	3
8	4	2	18	2	2
9	7	7	19	6	3
10	6	8	20	5	7

of positive association between the marginal random variables. In Appendix 7.2 we report the Matlab code for the estimation of the copula parameter in the case of complete sample.

A Kolmogovov-Smirnov test for the two marginal random variables X and Y doesn't reject the null hypothesis that the data come from a univariate exponential distributions with failure rate 0.16.

The empirical copula (Deheuvels, 1979) can be used to perform a goodness of fit test of the copula through the distance between a copula and the empirical one. Generally, the smallest distance to the reference copula implies the best fit. The comparison is defined by the Cramér von Mises statistics. Genest et al. (2009) use a parametric bootstrap procedure to obtain approximate p-values. The approximate p-value doesn't allow us to reject the null hypothesis that the data come from a MOC copula. For the computational aspect of the test see Genest *et al.* (2009).

A Matlab program was developed to construct the empirical copula (see Appendix 7.2). Figure 4 shows the IFM copula estimate and the empirical one.

For our second example we analysed data from Meintanis (2007). The data are represented in Tab. 7. It is the football (soccer) data, where at least one goal is scored by the home team and at least one goal is scored directly from a kick (penalty kick, foul kick or other direct kick) by any team. Let X be the time in minutes of the first kick goal scored by any team and let Y be the time in minutes of the first goal of any type scored by the home team. All events are possible $X > Y, X < Y$ and $X = Y$. Meintanis analysed the data using a MOBE distribution.

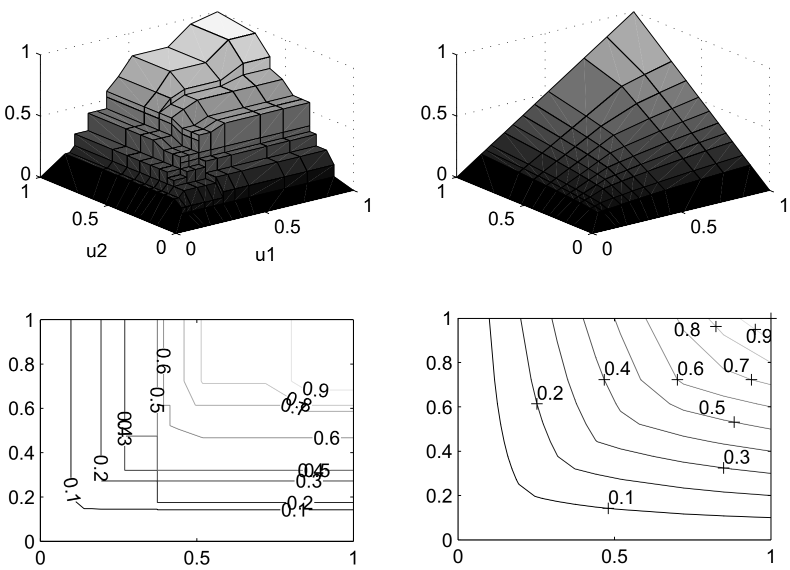

Figure 4: Empirical copula and IFM copula estimate

Kundu and Dey analyzed the data using a MOBWE distribution. We consider a MOBWE and we apply the proposed estimation procedure in order to estimate the parameters of the marginal Weibull distributions in (26) and the copula parameter. The values in the dataset have been divided by 100. The estimates of the shape and scale parameters of the marginal Weibull distributions of X and Y are $(2.121, 0.459)$ and $(1.421, 0.361)$, respectively. The estimate of the copula parameter is $\widehat{\theta} = 0.52$. The value of θ describes a mean grade of positive association between the marginal random variables. A Kolmogovov-Smirnov test for the two marginal random variables X and Y doesn't reject the null hypothesis that the data come from univariate Weibull distributions. The p-values of the test are 0.942 and 0.766, respectively.

The estimate of the copula parameter is $\widehat{\theta} = 0.710$. The value of θ describes a positive association between the marginal random variables. The approximate p-value of the goodness of fit test based on the Cramér von Mises statistic doesn't allow us to reject the null hypothesis that the data come from a MOC copula.

We have fitted the MOBE model also to the data set as suggested by Meintains. The marginal exponential distributions parameter estimates are $\lambda_X^* = 2.45$ and $\lambda_Y^* = 3.04$ and the θ parameter estimate is 0.729. The p-values of the Kolmo-

Table 7: Uefa Champions' League data for the years 2005-2006 and 2004-2005

2005 – 2006	X	Y	2004 – 2005	X	Y
Lyon-Real Madrid	26	20	Internazionale-Bremen	34	34
Milan-Fenerbahce	63	18	Real Madrid-Roma	53	39
Chelsea-Anderlecht	19	19	Man. United-Fenerbahce	12	12
Club Brugge-Juventus	66	85	Bayern-Ajax	51	28
Fenerbahce-PSV	40	40	Moscow-PSG	76	64
Internazionale-Rangers	49	49	Barcellona-Shakhtar	64	15
Panathinaikos-Bremen	8	8	Leverkusen-Roma	26	48
Ajax-Arsenal	69	71	Arsenal-Panathinaikos	16	16
Man. United-Benfica	39	39	Dynamo Kyiv-Real Madrid	44	13
Real Madrid-Rosemborg	82	48	Man. United-Sparta	25	14
Villarreal-Benfica	72	11	Bayern- M. TelAviv	55	11
Juventus-Bayern	66	62	Bremen-Internazionale	49	49
Club Brugge-Rapid	25	9	Anderlecht-Valencia	24	24
Olympiacos-Lyon	41	3	Panathinaikos-PSV	44	30
Internazionale-Porto	16	75	Arsenal-Rosenborg	42	3
Schalke-PSV	18	18	Liverpool-Olympiacos	27	47
Barcellona-Bremen	22	14	M. Tel-Aviv-Juventus	28	28
Milan-Schalke	42	42	Bremen-Panathinaikos	2	2
Rapid-Jouventus	36	52			

gorov-Smirnov test for the two marginal distributions are 0.0072 and 0.4465, respectively. It is clear that, although Meintanis suggests using MOBE, MOBVE is preferable in this case.

Finally, an example with Type-II censored data is analysed. We consider the data of 71 white male patients with diabetic retinopathy. For each patient, one eye was assigned to laser treatment while the other eye was assigned to follow up without treatment. Best corrected visual acuity was chosen as response variable for evaluating treatment. The failure times in days for the eyes of the 71 patients are reported in Table 3 of Csörgö and Welsh (1985). Some ties within pairs occur (10 out 71) so the Authors suggest that the MOBE is certainly a plausible model for the data.

We consider the censored sample in Fig. 3. We suppose terminating the experiment once 60 failures have been observed. The unobserved values, in one or both components, are set equal to 1519 days (the length of the experiment). For computational purposes, the values in the dataset have been divided by 100. The estimation of the exponential marginal parameters are $\lambda_X^* = 0.13$ and $\lambda_Y^* = 0.14$.

For the parameter estimation procedure using type-II censored data see Choen (1991). We observe that the mean failure times in days for the treated and untreated patients are 805 end 730, respectively. By the procedure described in Section 4.1 we find the estimate of the copula parameter, $\widehat{\theta} = 0.26$. For the computational aspects see Appendix 7.2.

Finally, we estimate the model with the complete sample. The estimates have values similar to the ones obtained with the censored sample: $\lambda_X^* = 0.133$, $\lambda_Y^* = 0.146$ and $\widehat{\theta} = 0.3$. The length of the experiment is 1939. Therefore, by using the censored sampling we obtained a considerable reduction of the length of the experiment and so a decrease in the cost of the experiment. However, it should be noted that the marginal distributions do not appear to be exponential: the Kolmogorov-Smirnov test for the marginal distributions rejects the null hypothesis that the data come from exponential distributions. Instead, applying marginal Weibull distributions we obtained appreciable results: the estimates of the shape and scale parameters are $(1.64, 8.45)$ and $(1.64, 7.68)$ and the associated p-values are 0.79 and 0.56. Therefore, a bivariate Weibull distribution (MOBWE) is preferable in this case.

CONCLUSIONS

In this work we have investigated inferential aspects of copula models. In particular we have considered the IFM method to estimate the parameters of bivariate survival distributions generated by the Marshal-Olkin copula. We found the estimators of MOC copula parameter and we presented also the log-likelihood function in the cases of both complete sample and censored sample (type-II censored sample). We analysed the asymptotic properties of the estimators by using Monte Carlo simulations. We presented good results both in the cases of complete sample and censored sample. We showed that our bias and mean square error are lower than the ones obtained through EM algorithm. Finally, several data sets were analysed. Development of asymptotic properties of the estimators not only in an empirical way is a topic of current research.

7. APPENDIX

7.1. ψ ESTIMATOR: COMPLETE AND CENSORED SAMPLE

We provide the maximum likelihood estimator (18) in the case of complete sample. The conditional log-likelihood function in (17) is

$$\begin{aligned} l(\psi|\underline{\hat{u}},\underline{\hat{v}}) &= k+(n_1+n_2)\ln[1-(1+\exp(-\psi))^{-1}]+n_3\ln[(1+\exp(-\psi))^{-1}]+ \\ &- (1-(1+\exp(-\psi))^{-1})(S_1+S_2)-(1+\exp(-\psi))^{-1}S_{\max}, \end{aligned}$$

where $S_1=\sum_{i=1}^{n}[-\ln(\hat{u}_i)]$, $S_2=\sum_{i=1}^{n}[-\ln(\hat{v}_i)]$ and $S_{\max}=\sum_i^n \max[-\ln(\hat{u}_i),-\ln(\hat{v}_i)]$. It can be simplified as

$$\begin{aligned} l(\psi|\underline{\hat{u}},\underline{\hat{v}}) &= k+(n-n_3)(-\psi)-n\ln(1+\exp(-\psi))+ \\ &- \exp(-\psi)(1+\exp(-\psi))^{-1}(S_1+S_2)-(1+\exp(-\psi))^{-1}S_{max}. \end{aligned}$$

By differentiating the log-likelihood function with respect to ψ we obtain

$$\begin{aligned} \frac{\partial l(\psi|\underline{\hat{u}},\underline{\hat{v}})}{\partial\psi} &= (-n+n_3)+n\frac{\exp(-\psi)}{(1+\exp(-\psi))}-\frac{\exp(-\psi)}{(1+\exp(-\psi))}S_{max}+ \\ &+ (S_1+S_2)\left[\frac{\exp(-\psi)}{(1+\exp(-\psi))^2}\right] \end{aligned}$$

Setting $\frac{\partial l(\psi|\underline{\hat{u}},\underline{\hat{v}})}{\partial\psi}=0$ we obtain

$$n_3\exp(-2\psi)-(n-2n_3-S_{\min})\exp(-\psi)-(n-n_3)=0,$$

where $S_{\min}=S_1+S_2-S_{\max}$.
By solving the previous equation with respect to $\exp(-\psi)$ we obtain two solutions

$$z_{1,2}=\frac{n-2n_3-S_1\pm\sqrt{n^2+S_{min}^2-S_{\min}(2n-4n_3)}}{2n_3}.$$

Since only the solution $z_1=\frac{n-2n_3-S_1+\sqrt{n^2+S_min^2-S_{\min}(2n-4n_3)}}{2n_3}$ has positive values, it is the unique accepted solution for $\exp(-\psi)$. Therefore, the unique solution of the optimization problem is

$$\hat{\psi}=-\ln\left[\frac{n-2n_3-S_{\min}+\sqrt{n^2+S_{\min}^2-S_{\min}(2n-4n_3)}}{2n_3}\right]. \qquad (27)$$

By calculating the second derivative we obtain that the solution is a maximum.

We provide the maximum likelihood estimator (21) in the case of censored sample. The use of the observations is described in Fig. 3. The conditional log-likelihood function in (20) is

$$\begin{aligned} l(\psi|\hat{u},\hat{v}) &= k+(m_1+m_2+r+s)\ln[1-(1+\exp(-\psi))^{-1}]+m_3\ln[(1+\exp(-\psi))^{-1}]+ \\ &- (1-(1+\exp(-\psi))^{-1})(S_1+S_2)-(1+\exp(-\psi))^{-1}S_{\max}, \end{aligned}$$

where

$$S_1=\sum_{i=1}^{m+r}\left[-\ln\left(\hat{u}_i\right)\right]+\left(n-m-r\right)w_{m:n},$$

$$S_2=\sum_{i=1}^{m+s}\left[-\ln\left(\hat{v}_i\right)\right]+\left(n-m-s\right)w_{m:n}$$

and $\mathrm{S}_{\max}=\sum_{i=1}^{m}\max\left[-\ln\left(\hat{u}_i\right),-\ln\left(\hat{v}_i\right)\right]+r\left(\mathrm{w}_{\mathrm{m:n}}\right)+s\left(\mathrm{w}_{\mathrm{m:n}}\right)+\left(n-m-r-s\right)\mathrm{w}_{\mathrm{m:n}}$.

It can be simplified as

$$\begin{aligned} l(\psi|\hat{u},\hat{v}) &= k+(m_1+m_2+r+s)(-\psi)-(m+r+s)\ln[(1+\exp(-\psi))]+ \\ &- \frac{\exp(-\psi)}{(1+\exp(-\psi))}(S_1+S_2)-(1+\exp(-\psi))^{-1}S_{\max}. \end{aligned}$$

By differentiating the log-likelihood function with respect to ψ we obtain

$$\begin{aligned} \frac{\partial l(\psi|\underline{\hat{u}},\underline{\hat{v}})}{\partial\psi} &= -(m_1+m_2+r+s)+(m+r+s)\frac{\exp(-\psi)}{(1+\exp(-\psi))}+ \\ &- \frac{\exp(-\psi)}{(1+\exp(-\psi))^2}S_{max}+(S_1+S_2)\left[\frac{\exp(-\psi)}{(1+\exp(-\psi))^2}\right] \end{aligned}$$

Setting $\frac{\partial l(\psi|\underline{\hat{u}},\underline{\hat{v}})}{\partial\psi}=0$ we obtain

$$m_3\exp(-2\psi)-(m+r+s-2m_3+S_{\min})\exp(-\psi)-(m+r+s-m_3)=0,$$

where $S_{\min}=S_1+S_2-S_{\max}$.

By solving the previous equation with respect to $\exp(-\psi)$ we obtain two solutions

$$z_{1,2}=\frac{m+r+s-2m_3-S_{\min}\pm\sqrt{(m+r+s-2m_3-S_{min})^2+4m_3(m+r+s-m_3)}.}{2m_3}$$

Since only the solution $z_1=\frac{m+r+s-2m_3-S_{\min}+\sqrt{(m+r+s-2m_3-S_{min})^2+4m_3(m+r+s-m_3)}}{2m_3}$ has positive values, it is the unique accepted solution for $\exp(-\psi)$. The unique solution of the optimization problem is

$$\hat{\psi}_c=-\ln\left[\frac{m+r+s-2m_3-S_{\min}+\sqrt{(m+r+S_{\min}-2m_3)^2+4m_3(m+r+s-m_3)}}{2m_3}\right].$$

By calculating the second derivative we obtain that the solution is a maximum.

Finally, we provide the maximum likelihood estimator (25) in the case of censored sample. The use of the observations is described in Fig. 2. The conditional log-likelihood function in (24) is

$$\begin{aligned} l(\psi|\hat{u},\hat{v}) &= k+(m_1+m_2)\ln[1-(1+e^{-\psi})^{-1}]+m_3\ln[(1+e^{-\psi})^{-1}]+ \\ &- (1-(1+e^{-\psi})^{-1})(S_1+S_2)-(1+e^{-\psi})^{-1}S_{\max}, \end{aligned}$$

where

$$S_1=\sum_{i=1}^{m}\left[-\ln\left(\hat{u}_i\right)\right]+\left(n-m\right)w_{m:n}$$

$$S_2=\sum_{i=1}^{m}\left[-\ln\left(\hat{v}_i\right)\right]+\left(n-m\right)w_{m:n}$$

and $S_{\max}=\sum_{i=1}^{m}\max\left[-\ln\left(\hat{u}_i\right),-\ln\left(\hat{v}_i\right)\right]+\left(n-m\right)w_{m:n}$. It can be simplified as

$$\begin{aligned} l(\psi|\hat{u},\hat{v}) &= k+(m_1+m_2)(-\psi)-m\ln[(1+e^{-\psi})]+ \\ &- \frac{e^{-\psi}}{(1+e^{-\psi})}(S_1+S_2)-(1+e^{-\psi})^{-1}S_{\max}. \end{aligned}$$

By differentiating the log-likelihood function with respect to ψ we obtain

$$\begin{aligned} \frac{\partial l(\psi|\underline{\hat{u}},\underline{\hat{v}})}{\partial\psi} &= -(m_1+m_2)+m\frac{\exp(-\psi)}{(1+\exp(-\psi))}+ \\ &- \frac{\exp(-\psi)}{(1+\exp(-\psi))^2}S_{max}+(S_1+S_2)\left[\frac{\exp(-\psi)}{(1+\exp(-\psi))^2}\right]. \end{aligned}$$

Setting $\frac{\partial l(\psi|\underline{\hat{u}},\underline{\hat{v}})}{\partial\psi}=0$ we obtain

$$m_3\exp(-2\psi)-(m-2m_3-S_{\min})\exp(-\psi)-(m-m_3)=0,$$

where $S_{\min}=S_1+S_2-S_{\max}$. By solving the previous equation we obtain two solutions for $\exp(-\psi)$

$$z_{1,2}=\frac{m-2m_3-S_{\min}\pm\sqrt{(m-2m_3-S_{min})^2+4m_3(m-m_3)}}{2m_3}.$$

Since only the solution $z_1=\frac{m-2m_3-S_{\min}+\sqrt{(m-2m_3-S_{min})^2+4m_3(m-m_3)}}{2m_3}$ has positive values, the unique solution of the optimization problem is

$$\hat{\psi}_c=-\ln\left[\frac{m-2m_3-S_{\min}+\sqrt{(m-2m_3-S_{\min})^2+4m_3(m-m_3)}}{2m_3}\right].$$

By calculating the second derivative we obtain that the solution is a maximum.

7.2. MATLAB CODES

This procedure describes the MOC parameter estimation having observed a complete sample. Let n be the sample size and u and v be the values of the two marginal cumulative distribution functions.

```
x1=data1;
x2=data2;
orx1=sort(x1);
orx2=sort(x2);
n3=0;
  for i=1:n
    if x1(i,1)==x2(i,1);
      n3=n3+1;
    end
  end
n33=n3;
u=; % vector of -ln of the first marginal cdf values
v=; %vector of -ln of the second marginal cdf values
smin=sum(min(u,v));
trasf=-log(n-2*n33-smin+
((n^ 2+smin\^ 2-smin*(2*n-4*n33))^ (1/2)))/(2*n33);
teta=1/(exp(-trasf)+1);
```

The following procedure describes the MOC parameter estimation having observed a censored sample (see Fig. 3).

```
The type-II censoring
w=max(x1,x2);
W=[x1,x2,w];
rowsort= sortrows(W,3);
k=; % the fraction of observed values
  for j=1:k
    d(j,:)=rowsort(j,:);
  end
  for j=k+1:n
    if rowsort(j,1)>=rowsort(k,3);
      d(j,1)=rowsort(k,3);
```

```
    else
     d(j,2)=rowsort(j,2);
    end
  end
u=; % vector of -ln of the first marginal cdf values
v=; % vector of -ln of the second marginal cdf values
m3=0;
  for i=1:k
    if x1(i,1)==x2(i,1);
      m3=m3+1;
    end
  end
m33=m3;
smin=sum(min(z1,z2));
trasf=-log((n-(n-k-r33-s33)-2*m33-smin)+((n-(n-k-r33-s33)-2*m33-smin)^ 2
+4*m33*(n-(n-k-r33-s33)-m33))^ (1/2))/(2*m33);
tetacen=1/(exp(-trasf1)+1);
```

The following procedure describes the empirical copula.

```
Function Fr=BECDF(X);
x1=X(:,1);
x2=X(:,2);
orx1=sort(x1);
orx1=sort(x2);
n=size(X);
  for s=1:n;
    for q=1:n
      h=0;
        for i=1:n
          if X(i,1)<=orx1(s,1)\& X(i,2)<=orx2(q,1)
            h=h+1;
          else h=h;
          end
        end
      H(:,q)=h;
    end
    HH(s,:)=H;
  end
```

```
HK=HH/n;
Fr=[orx1,orx2,HK];
X=[u,v];
o=BECDF(X);
```

Acknowledgement

The author wishes to thank professor Umberto Magagnoli and the Referees for their comments and suggestions.

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