

## MEASURING PROCESS CAPABILITY UNDER NON CLASSICAL ASSUMPTIONS: A PURPOSEFUL REVIEW OF THE RELEVANT LITERATURE

Claudio G. Borroni, Manuela Cazzaro, Paola M. Chiodini<sup>1</sup>

*Department of Statistics and Quantitative Methods, University of Milano-Bicocca, Milan, Italy*

**Abstract.** Since the 1980's the theory of process capability indices has been improved through a large number of indices. All the classical indices were originally designed to be used with normally distributed data; in addition, processes were supposed to be in control and centered on the specification interval. Thereafter, researchers realised that, if even one of the above properties is not met, the use of classical process capability indices can be misleading. In this paper, we review the literature concerning the consequences of such an incorrect use. More specifically, recent univariate indices are discussed in their ability to overcome the loss of normality or to face the presence of asymmetric specification limits. Sampling properties of such indices are considered as well. This review is far from giving an exhaustive list of all recent papers about process capability indices. The main focus is indeed on papers dealing with the loss of the above assumptions and particularly on those which have generated a relevant discussion.

**Keywords:** Univariate process capability indices, Non-normal data, Asymmetric tolerance limits, Estimation.

### 1. INTRODUCTION

Roughly speaking, a Process Capability Index (PCI) aims at measuring the ability of a process, which is usually assumed to be in statistical control and meet some specifications. To get into details, given that  $X$  is a characteristic of a process, you need to assess if  $X$  belongs to the interval  $(LSL, USL)$ ; whose extremes are known as *upper* and *lower specification limits*. As  $X$  is random, a PCI will be naturally based on the *probability* of  $X$  belonging to the specification interval or on the *mean deviation* of  $X$  from the specification limits. These criteria are often two equivalent aspects of a process. Consider, for instance, the classical *process capability ratio* (Kane, 1986):

$$C_p = \frac{USL - LSL}{6\sigma}, \quad (1)$$

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<sup>1</sup> Claudio G. Borroni, email: claudio.borroni@unimib.it

where  $\sigma^2$  denotes the variance of  $X$  (from now on,  $\mu = E(X)$  denotes the mean of the process, so that  $\sigma^2 = E(X - \mu)^2$ ). The index  $C_p$  compares the length of the specification interval with the actual range of  $X$  under the normality assumption. If  $X$  varies in a limited part of the specification range, i.e. if  $C_p$  takes a value higher than one, the process is likely to meet such specifications. Equivalently,  $C_p > 1$  is likely to be connected to a high expected proportion of conforming items or to a high *process yield*. This reasoning is the origin of the stronger criticism often moved to  $C_p$ , as the index can be meaningless if the specification interval is not centered around the mean  $\mu$ . When the mean moves away from the value  $M = (LSL + USL)/2$ , indeed, the process yield decreases even if  $C_p$  may remain very high. A classical proposal to overcome such a drawback is then to consider the two ratios

$$C_l = \frac{\mu - LSL}{3\sigma} \quad C_u = \frac{USL - \mu}{3\sigma} \quad (2)$$

and to define the index

$$C_{pk} = \min\{C_l, C_u\} = \frac{d - |\mu - M|}{3\sigma}, \quad (3)$$

where  $d = (USL - LSL)/2$ .

In order to relate a capability index to the process yield, researchers used to assume that  $X$  follows the normal distribution. It is now largely recognised that many process characteristics do not meet such an assumption. A particular effort, especially in the latest years, has then been paid to develop alternative indices based on different or no distributional assumptions at all. This subject will be the object of Section 2 of the paper. After specifying the distribution of the characteristic  $X$  (and possibly discussing its non-normality), the researcher may be tempted to measure the process capability directly by means of the expected proportion of non-conforming items. Nonetheless, it seems that, especially from the viewpoint of practitioners, the appeal of such indices like  $C_p$ , based on the mean departure of  $X$  from the specification limits (that is, essentially on the variability of  $X$ ), has never been attained by a direct measure of the process yield. This subject is very well discussed by Kotz and Johnson (2002).

In the 1980's, a new attention was paid, essentially by the Taguchi school, to the *target value*  $T$  as a criterion to build a capability index. Roughly speaking, a process capability analysis should encourage a producer not just to stay in the specification range, but also to reach a top-quality level. Indices like  $C_p$  and  $C_{pk}$ , which are independent of  $T$ , may instead give a good judgement on the process even if  $X$  strongly deviates from  $T$ , i.e. if  $E(X - T)^2$  takes large values. By

considering that  $E(X - T)^2 = \sigma^2 + (\mu - T)^2$ ,  $C_p$  in (1) can be modified as follows:

$$C_{pm} = \frac{USL - LSL}{6 \sqrt{\sigma^2 + (\mu - T)^2}} \quad (4)$$

(Hsiang and Taguchi, 1985). Notice that  $C_{pm}$  depends on the quantity  $(\mu - T)^2$ , which penalizes the value of the index if the process has a low *centring*, i.e. if its mean is far from the target. Usually  $T$  is around the midpoint  $M$  of the specification interval, which means that, when  $\mu$  is far from the target, the specification interval is also not centred at the mean. A good index should then also account for the corresponding process yield, as it was for  $C_{pk}$  in (3). By applying such a modification to  $C_{pm}$ , that is by combining  $C_{pk}$  with  $C_{pm}$ , a new index follows:

$$C_{pmk} = \frac{d - |\mu - M|}{3 \sqrt{\sigma^2 + (\mu - T)^2}} \quad (5)$$

(Pearn *et al.*, 1992). Of course,  $C_{pk}$ ,  $C_{pm}$  and  $C_{pmk}$  belong to the same family, as they are all modifications of the basic  $C_p$ , to which they reduce when  $\mu = M = T$ . Vännman (1995) suggests considering a general class:

$$C_p(u, v) = \frac{d - u|\mu - M|}{3 \sqrt{\sigma^2 + v(\mu - T)^2}}. \quad (6)$$

It results that  $C_p = C_p(0, 0)$ ,  $C_{pk} = C_p(1, 0)$ ,  $C_{pm} = C_p(0, 1)$  and  $C_{pmk} = C_p(1, 1)$ . However, different indices can be defined by choosing other values of  $u$  and  $v$ , like for the combination  $(u = 0, v = 4)$  suggested by the author herself. In a sense, such values reflect the “weight” the researcher assigns respectively to the yield and to the centring when measuring the capability of a process. The recent literature has emphasised some weaknesses of the  $C_p(u, v)$  class. Sometimes the specification interval happens not to be centred at the target, i.e.  $|T - M|$  may be not negligible. In such a situation, often referred to as *asymmetry of tolerances*, the yield and the centring may be conflicting criteria in the evaluation of the capability of a process, so that new suitable indices are required. This issue will be dealt with in Section 3.

Another class of PCIs stems from a different link with the process yield. If we denote by  $p$  the proportion of non-conforming items, it is easy to show that  $2\Phi(3C_p) - 1 = 1 - p$ , when  $X$  has a normal distribution with mean  $\mu = M$ , where  $\Phi(\cdot)$  denotes the standard-normal cumulative distribution function (cdf). Boyles (1994) proposes an index  $S_{pk}$  such that the same relation holds even if  $\mu \neq M$ :

$$S_{pk} = \frac{1}{3} \Phi^{-1} \left\{ \frac{1}{2} [\Phi(3C_l) + \Phi(3C_u)] \right\} = \frac{1}{3} S(3C_l, 3C_u). \quad (7)$$

Formula (7) shows that  $S_{pk}$  can be regarded as a “smooth” version of  $C_{pk}$  in (3), obtained (up to the multiplicative constant  $\frac{1}{3}$ ) by applying the function

$$S(x, y) = \Phi^{-1} \left\{ \frac{1}{2} [\Phi(x) + \Phi(y)] \right\} \quad x, y > 0, \quad (8)$$

instead of the function  $\min(x, y)$ , to the couple  $(3C_l, 3C_u)$ . To get another index potentially related both to the yield and to the centring, a “smooth” version of  $C_{pmk}$  in (5) can be considered, according to the function (8):

$$S_{pmk} = \frac{1}{3} S \left( \frac{\mu - LSL}{\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{USL - \mu}{\sqrt{\sigma^2 + (\mu - T)^2}} \right). \quad (9)$$

The index  $S_{pmk}$  will be referred to in Section 3 because, albeit with limits, it proves to be a good alternative to the  $C_p(u, v)$  family when tolerances are asymmetric.

An important issue regarding a PCI is its estimation. Sophisticated indices can indeed be useful tools, but their estimation is often cumbersome, so that the final value may still result in an inefficient measure of the real process capability. The advantages of an index need to be investigated also in terms of the simplicity of its estimation and/or of the availability of results about its sample distribution. In the estimation process, the researcher needs to face many choices: (i) single samples or multiple samples can be used; (ii) the distribution of the process can be left unspecified or a given model has to be chosen; (iii) point estimates or confidence intervals can be provided. The second issue has received a great attention, especially in the recent literature. Classical results about the estimation and the sample distribution of PCIs are usually based on the assumption of normality. We will give some details of these results in the following. Section 2, instead, aims at reviewing some recent results when the assumption of normality is not met.

The index  $C_p$  in (1) contains only the parameter  $\sigma$  to be estimated. If a single sample  $X_1, \dots, X_n$  of size  $n$  is given, a natural estimator  $\hat{C}_p$  of  $C_p$  will then simply replace  $\sigma$  by  $S = \sqrt{\sum_{i=1}^n (X_i - \bar{X})^2 / (n - 1)}$ , where  $\bar{X}$  denotes the sample mean. Chou and Owen (1989) obtained the probability density function (pdf) of  $\hat{C}_p$  (see also Pearn *et al.* (1992) and Kotz and Johnson (1993)). From this result  $\hat{C}_p$  is seen to be biased and to overestimate the actual value of  $C_p$ . Pearn *et al.* (1998) proposed then an unbiased minimum-variance estimator  $\tilde{C}_p$  which is also consistent, asymptotically efficient and such that  $\sqrt{n}(\tilde{C}_p - C_p)$  converges in distribution to  $N(0, C_p^2/2)$  (see also Chan *et al.* (1990)). Pearn *et al.* (1998) determined a confidence interval for  $C_p$ , (see also Chou *et al.* (1990) and Heavlin (1998) for alternative confidence limits for  $C_p$ ). Kirmani *et al.* (1991) considered the estimation

of  $C_p$  and its properties based on  $m$  multiple samples. They proposed an unbiased and consistent estimator,  $\hat{C}_p^*$ , and obtained its pdf. They also proposed a lower confidence bound on  $C_p$ . More recently, Pearn *et al.* (2003) investigated some statistical properties of  $\hat{C}_p^*$  and showed that  $\hat{C}_p^*$  is asymptotically efficient and such that  $\sqrt{mn}(\hat{C}_p^* - C_p)$  converges in distribution to  $N(0, C_p^2/2)$ .

The natural estimator  $\hat{C}_{pk}$  of  $C_{pk}$  in (3), based on a single sample, is obtained by replacing the process mean  $\mu$  and the process standard deviation  $\sigma$  with the sample mean  $\bar{X}$  and  $S$ , respectively. Pearn *et al.* (1993) derived the  $r$ -th moment of  $\hat{C}_{pk}$  (see also Zhang *et al.*, 1990; Pearn *et al.*, 1992; Kotz and Johnson, 1993) and they showed that  $\hat{C}_{pk}$  is a biased estimator of  $C_{pk}$ , even if its bias tends to zero and it is consistent. The pdf of  $\hat{C}_{pk}$  is given by Chou and Owen (1989) and, in a simplified way, by Pearn *et al.* (1999). Lin and Pearn (2003) derived the cdf of  $\hat{C}_{pk}$  in a different form, even if its asymptotic limit was earlier obtained by Chan *et al.* (1990). There are many contributions in the literature concerning confidence intervals for  $C_{pk}$ , due to the difficulties of the distribution of  $\hat{C}_{pk}$ , (see, among others, Chou and Owen, 1989; Chou *et al.*, 1990; Zhang *et al.*, 1990; and Nagata and Nagahata, 1994). Li *et al.* (1990) studied the distribution of the estimator of  $C_{pk}$  based on multiple samples.

The index  $C_{pm}$  in (4) requires the estimation of two parameters,  $\mu$  and  $\sigma$ . Chan *et al.* (1988) and Boyles (1991) proposed two different, asymptotically equivalent, estimators of  $C_{pm}$  ( $\tilde{C}_{pm}$  and  $\hat{C}_{pm}$  respectively), both based on a single sample. The estimator  $\tilde{C}_{pm}$  uses  $S$ , while the estimator  $\hat{C}_{pm}$  is based on  $S_n = \sqrt{\frac{n-1}{n}}S$ . Chan *et al.* (1988) derived the pdf of  $\tilde{C}_{pm}$  (see also Pearn *et al.*, 1992) and showed that their estimator is asymptotically unbiased and weakly consistent. Under normality,  $\hat{C}_{pm}$  is the MLE of  $C_{pm}$  and it is generally positively biased. Kotz and Johnson (1993) derived the finite  $r$ -th moment of  $\hat{C}_{pm}$  for  $r < n$ , while Boyles (1991) and Pearn *et al.* (1992) derived its pdf (see also Chen *et al.*, 1999 and Vännman and Kotz, 1995b). The cdf of Boyles' estimator  $\hat{C}_{pm}$  is also obtained by Vännman and Kotz (1995b) and by Pearn and Lin (2003) in different forms. Wright (1992) derived an explicit common form for the cdfs of both  $\tilde{C}_{pm}$  and  $\hat{C}_{pm}$ . The asymptotic distribution of  $\hat{C}_{pm}$  is obtained by Chan *et al.* (1990). Several methods have been suggested in the literature dealing with the construction of approximate lower confidence bounds of  $C_{pm}$  (see, among others, Boyles, 1991 and Chan *et al.*, 1990). As regards multiple samples, several estimators and of  $C_{pm}$  are available; see Zhang (2001), Vännman and Hubele (2003) and Pearn Shu (2003).

The natural estimator of  $C_{pmk}$  in (5),  $\hat{C}_{pmk}$ , was proposed by Pearn *et al.*

(1992) by using  $\bar{X}$  and  $S_n$  for  $\mu$  and  $\sigma$  respectively. The distribution of  $\hat{C}_{pmk}$  as long as its  $r$ -th moment were also derived. From these results  $\hat{C}_{pmk}$  is seen to be positively biased. The asymptotic distribution of  $\hat{C}_{pmk}$  was investigated by Chen and Hsu (1995) who showed that the estimator is consistent, asymptotically unbiased and, provided the fourth moment is finite, asymptotically normal. Wu and Liang (2010) corrected a critical error for the asymptotic distribution presented in Chen and Hsu (1995) when the population mean is not the midpoint of the specification interval. The pdf of  $\hat{C}_{pmk}$  has been explicitly obtained, in a rather complicated expression, by Wright (1998); see also Vännman and Kotz (1995b). Following Vännman (1997a) and Pearn and Lin (2002) the cdf and pdf of the estimator  $\hat{C}_{pmk}$  may be alternatively expressed. Pearn *et al.* (2001) obtained another simpler form of the pdf of  $\hat{C}_{pmk}$ . Following the asymptotic distribution obtained by Chen and Hsu (1995) (see also Wu and Liang (2010)), considerations on confidence intervals of  $C_{pmk}$  can be properly drawn. The estimation of  $C_{pmk}$  based on multiple samples was addressed by Vännman and Hubele (2003) and by Pearn and Shu (2003).

For the general class  $C_p(u, v)$  in (6), Vännman (1995) proposed two alternative estimators, that differ in the way the variance  $\sigma^2$  is estimated. In the same paper, the expected values, the variances and the mean square errors of the proposed estimators were derived. Note that all estimators are biased. Vännman and Kotz (1995b) provided an explicit form of the pdf of the family of estimators. Simplified expressions were provided by Vännman (1997a). The asymptotic expected value and mean square error were derived in Vännman and Kotz (1995a). The asymptotic distribution of  $\hat{C}_p(u, v)$ , the estimator using the sample variance  $S^2$ , was obtained independently in two different papers, Chen (1997) and Lin (2004), provided that the fourth moment is finite. Vännman and Hubele (2003) considered the estimation of the family  $C_p(u, v)$  based on multiple samples.

The above review of classical PCIs can be extended by considering the multivariate case and further inferential problems, like hypothesis testing. In this paper, we confine ourselves to considering univariate processes and considering point estimation of PCIs. Even under such a limitation, the recent literature on the subject is quite large. As above mentioned, we will focus on such papers dealing with the loss of the assumptions upon which classically PCIs are based: the normality of the process and the symmetry of the specification limits with respect to the target. The present work is hence intended to be a purposive review of the related literature and, in this sense, the reader is to be warned that an exhaustive list of all recent papers about PCIs will not be provided. To this aim, refer to other recent

reviews about PCIs, like the general reviews by Kotz and Johnson (2002, with discussion), by Anis (2008) and by Wu *et al.* (2009), the references in Spiring *et al.* (2003) and by Yum and Kim (2011) or the encyclopedia by Pearn and Kotz (2006).

This paper is organised as follows. Section 2 reviews the recent literature about PCIs when the process characteristic follows a non-normal distribution. Section 3 deals with the literature about asymmetric tolerances. A short conclusion and some direction for future research are finally provided in Section 4.

## 2. NON-NORMAL PROCESSES

As revealed in the Introduction, an efficient estimation of a PCI rests on very restrictive assumptions: observations should be independent, the process should be stable, in control and the output should be normal. The most restrictive assumption is normality. To this end, in the last years many authors investigated the effects of the departure from this assumption. As above referred, a strong relation usually connects the value of a process capability index and the proportion of non-conforming items  $p$ . It is well-known that if the process is in control and the output is normally distributed, with a process centred at the middle point of the specification interval, there is a relation between the value of the capability indices in (1) and in (3) and  $p$ , i.e. if  $C_p = C_{pk} = 1.0$  then  $p$  is 0.27%. Symmetry alone is not enough to guarantee this relation. A symmetric distribution with heavy tails will indeed give a larger value of  $p$ . Pearn *et al.* (1992) underline that the value of  $C_{pk}$  is useful just to give a limit to the proportion of non-conforming items, which can never be larger than  $2\Phi(-3C_{pk})$ . The same relation is true also for the indices in (4) and (5): the probability of non-conformance is never larger than  $2\Phi(-3C_{pm})$  and  $2\Phi(-3C_{pmk})$ , respectively. Ruczinski (1985) provides a table showing that the same value of  $C_{pm}$  can be associated with different values of  $p$ . It is known that the relation between the values of a PCI and  $p$  gets weaker as the index is more sophisticated even if invalid results may already be obtained when classical indices are estimated by non-normal data. In particular  $C_p$  is very sensitive to skewness. It seems that the more the distribution is skewed the more the estimate of the index is biased especially for small sample size. It is worthwhile to note that  $C_{pm}$  can be applied to assess process capability for both normal and non-normal distributions since it is not distribution sensitive, as highlighted in Spiring *et al.* (2002), even if the underlying distribution may impact inference.

The next sub-sections will focus on three approaches which are mainly known in the literature to deal with departures from normality when measuring process

capability: a) transforming non-normal data to normality so that normal-based PCIs can be applied; b) fitting data by a model so that quantile-related PCIs can be evaluated; c) developing non-quantile-based indices applicable to non-normal distributions.

## 2.1. DATA TRANSFORMATIONS

A way to deal with non-normal data is to transform them in order to get data either normal or, at least, closer to normal. Classical indices can then be used on transformed data as if they were the original data. Two classical methods are Box-Cox's and Johnson's transformations.

Box and Cox (1964) identify the following transformation for a positive variable  $X$ :

$$X^{(\lambda)} = \begin{cases} \frac{X^\lambda - 1}{\lambda} & \text{for } \lambda \neq 0, \\ \ln X & \text{for } \lambda = 0. \end{cases} \quad (10)$$

The value of the parameter  $\lambda$  is estimated by a two-step maximum likelihood method. First a value of  $\lambda$  from a pre-assigned range is collected. Then the expression  $L_{\max}(\lambda) = -\frac{1}{2} \ln \hat{\sigma}^2 + (1 - \lambda) \sum_{i=1}^n \ln X_i$  is evaluated for all  $\lambda$ 's. The estimate of  $\hat{\sigma}^2$  for a fixed  $\lambda$  is  $\hat{\sigma}^2 = S(\lambda)/n$ , where  $S(\lambda)$  is the residual sum of squares in the analysis of variance of  $X^{(\lambda)}$ . After calculating  $L_{\max}(\lambda)$  for several values of  $\lambda$  within the chosen range, the plot of  $L_{\max}(\lambda)$  is drawn and the value that maximises the function is obtained. Then the corresponding PCIs can be estimated from the mean and the standard deviation based on transformed data.

Johnson (1949) introduces a system of distributions similar to the Pearson's system. The proposed system contains three families of distributions all defining a suitable transformation to generate normal data, of the form  $z = \gamma + \eta \tau(x; \lambda, \varepsilon)$ , where  $z$  is a standard normal variable and  $\gamma$ ,  $\eta$  and  $\tau(x; \lambda, \varepsilon)$  are chosen to cover a wide range of possible shapes. More specifically, the function  $\tau(x; \lambda, \varepsilon)$  may take one of the following forms:

$$\tau(x; \lambda, \varepsilon) = \sinh^{-1} \left( \frac{x - \varepsilon}{\lambda} \right), \quad (11)$$

$$\tau(x; \lambda, \varepsilon) = \ln \left( \frac{x - \varepsilon}{\lambda + \varepsilon - x} \right), \quad (12)$$

$$\tau(x; \lambda, \varepsilon) = \ln \left( \frac{x - \varepsilon}{\lambda} \right). \quad (13)$$

The function (11) is used if data come from unbounded distributions, like the  $t$  distribution; (12) if data come from bounded distributions, like the gamma and the beta distributions; (13) if data comes from skewed distributions, like the log-normal distribution.

**2.2 QUANTILE-BASED PROCESS CAPABILITY INDICES**

As the indices of the class  $C_p(u, v)$  in (6) are not adequate for non-normal processes, Pearn and Chen (1997) introduce the family of indices  $C_{Np}(u, v)$ . Let  $x_\alpha$  be the  $100\alpha$ -th percentile of the underlying distribution and  $x_{0.5}$  its median. The indices are:

$$C_{Np}(u, v) = \frac{d - u|x_{0.5} - M|}{3 \sqrt{\left[\frac{x_{0.99865} - x_{0.00135}}{6}\right]^2 + v(x_{0.5} - T)^2}} \quad (u, v \geq 0). \quad (14)$$

The quantity  $\frac{x_{0.99865} - x_{0.00135}}{6}$  is then set to mimic the related property of the normal distribution. The mean is replaced by the median that is a more robust measure of central tendency especially for skew distributions with long tails. By changing the values of  $(u, v)$  it is possible to obtain the four principal generalized indices for non-normal distributions:

$$C_{Np} = \frac{USL - LSL}{x_{0.99865} - x_{0.00135}} \quad (15)$$

$$C_{Npk} = \frac{\min(USL - x_{0.5}, x_{0.5} - LSL)}{\left[\frac{x_{0.99865} - x_{0.00135}}{6}\right]} \quad (16)$$

$$C_{Npm} = \frac{USL - LSL}{6 \sqrt{\left[\frac{x_{0.99865} - x_{0.00135}}{6}\right]^2 + (x_{0.5} - T)^2}} \quad (17)$$

$$C_{Npmk} = \frac{\min(USL - x_{0.5}, x_{0.5} - LSL)}{3 \sqrt{\left[\frac{x_{0.99865} - x_{0.00135}}{6}\right]^2 + (x_{0.5} - T)^2}}. \quad (18)$$

Notice that, when ranking the four indices according to their sensibility to departures from the target of the process median, the same result is obtained for classical indices ranked according to departures of the mean from the target. In decreasing order:  $C_{Npmk} > C_{Npm} > C_{Npk} > C_{Np}$  and  $C_{pmk} > C_{pm} > C_{pk} > C_p$ , respectively. Furthermore, if the process is on-target so that  $x_{0.5} = T$  all the indices reduce to  $C_{Np} = C_{Npk} = C_{Npm} = C_{Npmk} = d/3\sigma''$ , where  $\sigma'' = (x_{0.99865} - x_{0.00135})/6$ . Moreover, if the underlying distribution is normal,  $x_{0.5} = \mu$  and  $\sigma'' = \sigma$  and the generalised indices  $C_{Np}(u, v)$  reduce to the basic  $C_p(u, v)$  indices in (6).

Pearn and Kotz (1994) and Pearn and Chen (1995) apply a related method, originally proposed by Clements (1989), to obtain a version of  $C_p(u, v)$  useful for non-normal Pearson's distribution. The indices obtained by such an approach are:

$$C'_{Np}(u, v) = (1 - u) \times \frac{USL - LSL}{6 \sqrt{\left[\frac{F_{0.99865} - F_{0.00135}}{6}\right]^2 + v(F_{0.5} - T)^2}} + u \times \min \left\{ \frac{USL - x_{0.5}}{3 \sqrt{\left[\frac{F_{0.99865} - F_{0.5}}{3}\right]^2 + v(F_{0.5} - T)^2}}, \frac{USL - F_{0.5}}{3 \sqrt{\left[\frac{F_{0.5} - F_{0.00135}}{3}\right]^2 + v(F_{0.5} - T)^2}} \right\}, \quad (19)$$

where  $F_\alpha$  denotes the  $100\alpha$ -th percentile computed on standardized data fitted by a suitable Pearson's curve. By setting  $(u, v) = (0, 0); (0, 1); (1, 0); (1, 1)$  it is possible to obtain respectively the four generalisations of the basic indices for non-normal distributions. Moreover, if the distribution is normal, then  $F_{0.5} = \mu$ ,  $F_{0.99865} - F_{0.00135} = 6\sigma$ ,  $F_{0.99865} - F_{0.5} = 3\sigma$ ,  $F_{0.5} - F_{0.00135} = 3\sigma$ ; clearly the generalised indices  $C'_{Np}(u, v)$  reduce to the basic indices  $C_p(u, v)$  in (6).

Chang and Lu (1994) obtain the estimators  $\hat{C}_{Np}(u, v)$  of the family  $C_{Np}(u, v)$  in (14) by substituting the percentile  $x_{0.99865}$ ,  $x_{0.00135}$  and  $x_{0.5}$  with:

$$\begin{aligned} \hat{P}_{0.99865} &= X_{(\lfloor R_1 \rfloor)} + (R_1 - \lfloor R_1 \rfloor)(X_{(\lfloor R_1 \rfloor + 1)} - X_{(\lfloor R_1 \rfloor)}), \\ \hat{P}_{0.00135} &= X_{(\lfloor R_2 \rfloor)} + (R_2 - \lfloor R_2 \rfloor)(X_{(\lfloor R_2 \rfloor + 1)} - X_{(\lfloor R_2 \rfloor)}), \\ \hat{P}_{0.5} &= X_{(\lfloor R_3 \rfloor)} + (R_3 - \lfloor R_3 \rfloor)(X_{(\lfloor R_3 \rfloor + 1)} - X_{(\lfloor R_3 \rfloor)}); \end{aligned} \quad (20)$$

where  $R_1 = (99.865n + 0.135)/100$ ,  $R_2 = (0.135n + 99.865)/100$  and  $R_3 = (n + 1)/2$ . Note that  $\lfloor R \rfloor$  denotes the integer less than or equal to  $R$  and  $X_{(i)}$  is the  $i$ -th order statistic. In a similar way, the Chang and Lu method used in association with the Clements one, allows to obtain the estimators  $\hat{C}'_{Np}(u, v)$  of the class of indices  $C'_{Np}(u, v)$  in (19).

### 2.3. NON-QUANTILE-BASED PROCESS CAPABILITY INDICES

Wright (1995) proposes the index  $C_s$ , which is basically an adaption of  $C_{pmk}$  in (5), involving a skewness term in the denominator:

$$C_s = \frac{d - |\mu - T|}{3 \sqrt{\sigma^2 + (\mu - T)^2 + \left|\frac{\mu_3}{\sigma}\right|}} \quad (21)$$

where  $\mu_3 = E[(X - \mu)^3]$  is the third central moment and  $\frac{\mu_3}{\sigma}$  is a measure of skewness. The modification is proposed as  $C_{pmk}$  is too sensitive to changes in the shape

of the distribution, particularly in its skewness. It is worthwhile to note that  $C_s$  is applicable to processes whose mean may not be centred between the specification limits (as for  $C_{pmk}$  in (5)); furthermore it has the advantage that the specification interval does not need to be centred at the target. Note that the value of the index reduces in presence of asymmetry. The index, however, cannot be related to the process yield. Wright (1995) proposes a natural estimator  $\hat{C}_s$  of  $C_s$ :

$$\hat{C}_s = \frac{d - |\bar{X} - T|}{3 \sqrt{\frac{1}{n} \sum_{i=1}^n (X_i - T)^2 + \left| \frac{n^2 m_3}{(n-1)(n-2)} \times \left( \frac{n}{n-1} \times \frac{m_2}{c_4^2} \right)^{-1/2} \right|}} \quad (22)$$

where  $m_r = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^r$  are the  $r$ -th sample central moments and where  $c_4 = \left[ \frac{2}{n-1} \right]^{1/2} \times \Gamma\left(\frac{n}{2}\right) \Gamma\left[\frac{n-1}{2}\right]^{-1}$  is a correcting factor for bias. Wright (1995) studies the bias and the variance of  $\hat{C}_s$  by simulation for normally distributed processes, revealing that  $\hat{C}_s$  performs better than  $\hat{C}_{pmk}$  when significant shifts in the mean occur. Pearn and Chang (1997) extend Wright's simulation involving skewed distributions as chi-square, lognormal and Weibull distributions. They highlight that the percentage bias of the estimator increases as the coefficient  $|\mu_3/\sigma|$  increases.

Chen and Kotz (1996) propose a revised version of  $C_s$ , involving a multiplier  $\gamma > 0$  before  $|\frac{\mu_3}{\sigma}|$ . Different values of  $\gamma$  lead to different optimality requirements. On the basis of some simulations, Chen and Kotz (1996) investigate the consistency and asymptotics of  $\hat{C}_s$ , revealing that it is asymptotically really sensitive to skewness and highlighting that the skewness correcting factor contributes additively to the overall variability. Note that Wright (1995) and Chen and Kotz (1996) both agree on remarking that the asymptotic distribution of  $\hat{C}_s$  is Normal when  $\mu \neq M$  and  $\mu_3 \neq 0$ . This consideration leads to choose  $\hat{C}_s$  for process capability assessment for large sample sizes, regardless the underlying distribution. Note that  $\hat{C}_s$  performs at its best under the following conditions:  $\mu \neq M$ ,  $\mu \neq T$ ,  $\mu_3 \neq 0$ . Finally, bootstrap confidence intervals for  $C_s$  are given in Han *et al.* (2000).

As above remarked, the fundamental criticism moved to  $C_s$  is the lack of a relationship between the index and  $p$ , the proportion of non-conforming items. Magagnoli and Chiodini (2001) investigate how to choose  $\gamma$  in the above modification proposed by Chen and Kotz (1996), in order to establish a relation between  $p$  and the value of  $C_s$ , similarly to the one valid for classical indices under normality condition.

Another branch of new PCIs, to be used with possibly skewed distributions is essentially based on two methods: the Weighted Variance (WV) and the Weighted

Standard Deviation (WSD) methods. Both of them are based on the idea that a skewed distribution can be divided into a lower and an upper part, according to the value of the process mean, modeled as two normal distributions with the same mean  $\mu$  but different variances or standard deviations,  $\sigma_L$  and  $\sigma_U$ . Note that in the WV method,  $\sigma_L$  and  $\sigma_U$  are obtained as  $\sigma_L = \sqrt{(1-P)} \cdot \sigma$  and  $\sigma_U = \sqrt{P} \cdot \sigma$ , where  $P = Pr(X \leq \mu)$ , and that in the WSD method  $\sigma_L$  and  $\sigma_U$  are obtained as  $\sigma_L = (1-P) \cdot \sigma$  and  $\sigma_U = P \cdot \sigma$ . Furthermore, note that  $\sigma_L$  and  $\sigma_U$  represent the degrees of dispersions of the upper and lower sides from  $\mu$ , respectively. Therefore, the variances/standard deviations can be estimated separately. Simulation studies indicate that these methods are fairly consistent in estimating process non-conformance rate for some non-normal distributions.

Some new PCIs are proposed by Bai and Choi (1997) based on the WV approach. In particular, they propose the generalisation of  $C_p$  defined in (1) as:

$$C_p^{WV} = \min \left( \frac{USL - LSL}{6\sigma_U}, \frac{USL - LSL}{6\sigma_L} \right). \quad (23)$$

It is worth noting that, if  $\sigma_U = \sigma_L = \sigma$ ,  $C_p^{WV} = C_p$ . Bai and Choi (1997) propose also the indices:

$$C_{pk}^{WV} = \min \left( \frac{USL - \mu}{3\sigma_U}, \frac{\mu - LSL}{3\sigma_L} \right) \quad (24)$$

$$C_{pm}^{WV} = \min \left( \frac{USL - T}{3\sigma_U^T}, \frac{T - LSL}{3\sigma_L^T} \right) \quad (25)$$

$$C_{pmk}^{WV} = \min \left( \frac{USL - \mu}{3\sigma_U^T}, \frac{\mu - LSL}{3\sigma_L^T} \right), \quad (26)$$

where  $\sigma_j^T$ ,  $j = U, L$  are the upper and lower standard deviations from the target. Note that  $\sigma_L^T$  and  $\sigma_U^T$  are obtained as  $\sigma_L^T = \sqrt{(1-P_T)} \cdot \sigma$  and  $\sigma_U^T = \sqrt{P_T} \cdot \sigma$  where  $P_T = Pr(X \leq T)$ .

Wu *et al.* (1999) propose a new WV-based PCI, corresponding to  $C_p$ , using a different approximation of the length of the  $\pm 3\sigma$  limits:

$$\bar{C}_p^{WV} = \frac{USL - LSL}{(3 \cdot \sigma_L + 3 \cdot \sigma_U)}. \quad (27)$$

In a similar way, Wu *et al.* (1999) propose a generalisation of the normality-based PCIs,  $C_{pk}$ ,  $C_{pm}$ ,  $C_{pmk}$  (defined respectively in (3), (4) and (5)).

Chang *et al.* (2002) propose the analogous to  $C_p$  and  $C_{pk}$  for skewed population based on the WSD method, which improves the performance of the ones

proposed in Bai and Choi (1997). When the underlying distribution is symmetric, all the above indices reduce to standard PCIs.

All authors propose to estimate the above indices by replacing  $\mu$ ,  $\sigma$ ,  $P$  and  $P_T$  with the sample mean  $\bar{X}$ , the sample standard deviation, the number of observations less than or equal to  $\bar{X}$  and the number of observations less than or equal to  $T$ , respectively. Currently, only point estimates are provided for PCIs based on the VW and WSD methods as it is very difficult to derive the sample distribution of their estimators and hence to construct confidence intervals.

Johnson *et al.* (1994) propose a “flexible” PCI  $C_{jkp}$ , that is essentially a modification of  $C_{pm}$  in (4), which takes into account possible differences in variability above and below the target value. Furthermore, the  $C_{jkp}$  index is sensitive to the shape of the process distribution. It is defined as:

$$C_{jkp} = \min \left( \frac{USL - T}{3\sqrt{2}W_+}, \frac{T - LSL}{3\sqrt{2}W_-} \right), \quad (28)$$

where  $W_+^2 = E_{X>T}[(X - T)^2]$ , and  $W_-^2 = E_{X<T}[(X - T)^2]$ . The multiplier  $\sqrt{2}$  in the denominator is due to the fact that if  $X$  has a symmetric distribution with variance  $\sigma^2$  and expected value  $T$ , then  $W_+^2 = W_-^2 = \frac{\sigma^2}{2}$ . The estimator of  $C_{jkp}$  is:

$$\hat{C}_{jkp} = \min \left( \frac{USL - T}{3\sqrt{2}\sqrt{\frac{S_+^2}{n}}}, \frac{T - LSL}{3\sqrt{2}\sqrt{\frac{S_-^2}{n}}} \right), \quad (29)$$

where  $S_+^2 = \sum_{X_i>T}(X_i - T)^2$  and  $S_-^2 = \sum_{X_i<T}(X_i - T)^2$ . Note that  $S_+$  and  $S_-$  are unbiased estimators of  $W_+^2$  and  $W_-^2$ , respectively. Franklin and Wasserman (1994) investigate the distribution of  $\hat{C}_{jkp}$  using bootstrap simulation methods in order to determine the 95% lower confidence limits for this index and they show that  $C_{jkp}$  is much more robust with respect to other indices when data come from skewed processes.

It seems that the further research on PCIs for non-normal data is, by this time, only devoted to compare via simulations the performances of the different PCIs.

Tang and Than (1999) review several methods that are chosen for performance comparison in their ability to handle non-normality in PCIs. The comparison is done through simulating Weibull and lognormal data. They consider: the  $C_s$  Wright’s index in (21), the WV  $C_p$  index (in (23)) proposed by Bai and Choi (1997), the  $C_p$  and  $C_{pk}$  indices revised under Clements’ approach (see (19)), the  $C_p$  and  $C_{pk}$  indices calculated on the transformed data using Box-Cox transformation (10) and various Johnson’s transformations (unbounded, bounded and

lognormal) as reported in (11)–(13). The fraction of non-conforming items is fixed a priori by using suitable specification limits, and PCIs computed using various methods are then compared with a target value. The authors highlight that the performance of the transformation methods (Clements, Box-Cox, Johnson) is consistently better than that of the non-transformation methods for all the different underlying non-normal distributions, at least for large sample sizes, with one exception. Clements' method perform worse than the other two transformation methods, in particular in the case of the Weibull distribution. As the Box-Cox transformation is consistently superior in performance throughout, the authors conclude that it is the preferable method for handling non-normal data whenever a computer-assisted analysis is available. Furthermore, the non-transformation methods are found to be inadequate in capturing the capability of the process, except when the underlying distribution is close to normal. Generally speaking, one can conclude that the performance of a method depends on its capability to capture the tail behaviour of the underlying distributions.

Wu and Swain (2001) compare, through Monte Carlo simulation, the performances of some capability indices when processes are non-normally distributed. In particular, they consider the four classical indices  $C_p$ ,  $C_{pk}$ ,  $C_{pm}$ ,  $C_{pmk}$  revised under the Clements' method (see (19)), the Johnson-Kotz-Pearn  $C_{jkp}$  index (28) and the WV indices proposed by Wu *et al.* (1999) (see (27)). The Johnson's family of distributions is used to generate a variety of non-normal data to evaluate the sensitivity of different PCIs to a given extent of the process yield. After fixing a specific value of the percentage outside the specification limits, 14 different types of non-normal processes are considered: unbounded, bounded distributions as well as lognormal distributions. Wu and Swain (2001) conclude that for symmetric bounded and unbounded processes, the Clements' method performs best. For skewed bounded processes, none of the three methods seem to perform well. In addition, for skewed unbounded processes, Clements' method is misleading. The WV-based PCIs perform well in skewed unbounded and lognormal distributions compared to the Clements' method and to  $C_{jkp}$ .

As far as  $C_p$  and  $C_{pk}$  are concerned, Chang *et al.* (2002) compare via Monte Carlo simulation their WSD indices with classical PCIs, and with Clements' and Bai-Choi WV PCIs (see, respectively, (19) and (23)–(24)), when the distribution is normal, Weibull, lognormal and gamma. These models represent a wide variety of shapes, from symmetric to highly skewed. They highlight that the WSD method outperforms the other methodologies like the WV and the Clements' method, and also that its accuracy is better when the range of the specification limits is about

$6\sigma$ . In particular, the indices WSD proposed by Chang *et al.* (2002) perform better than other PCIs for skewed populations.

### 3. ASYMMETRIC TOLERANCES

Classical process capability indices are not reliable when tolerances are asymmetric with respect to the target. Asymmetric tolerances may occur due to different reasons. First, the customer is often more inclined to accept deviations from the target in one direction as opposed to the other, even though some symmetric tolerances may be initially imposed; the producer may then realize that the process will not be able to follow the required specifications and asks for an expansion of one side of the tolerances only, which raises greater problems. Another source of asymmetry of tolerances may be an intrinsic skewness of the process. In many cases, indeed, tolerances and targets are specified on the basis of chosen percentiles of the process distribution. For instance, if the customer is willing to accept a  $p\%$  of non-conforming items,  $LSL = x_{p/2}$ ,  $T = x_{0.5}$  and  $USL = x_{1-p/2}$  is likely to be set, where  $x_\alpha$  denotes the  $100\alpha$ -th percentile of the process distribution; however, if such a distribution is skewed,  $x_{0.5} - x_{p/2} \neq x_{1-p/2} - x_{0.5}$ , consequently that tolerances are not symmetric. Finally, a transformation of data to reduce them to normality can cause asymmetric tolerances. Suppose that a specification interval  $(USL, LSL)$  and a target  $T$  are fixed accordingly to a process characteristic  $X$ , which has however a non-normal distribution. The researcher may then apply a transformation  $Y = g(X)$  so that  $Y$  is approximately Normal, but the interval  $(g(USL), g(LSL))$  is likely not to be symmetric in respect to  $g(T)$ .

Some generalisations of the classical indices were formerly proposed by modifying the true specification limits to get a symmetric interval. This change can be done, for instance, by shifting the limit which is more distant from the target value  $T$ , i.e. by considering the new interval  $(T \pm d^*)$ , where  $d^* = \min\{USL - T, T - LSL\}$ . If such a modification is applied to the general Vännman family in (6), one gets

$$C_p^*(u, v) = \frac{d^* - u|\mu - T|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}. \quad (30)$$

See Kane (1986), Chan *et al.* (1988) and Vännman (1997b) for further details and developments. Today, it is well known that such a class of indices may understate the real process capability, due to the actual restriction of the specification limits to a proper subset of the original interval (Boyles, 1994). For instance, consider a process which is centred on  $M = (USL + LSL)/2$ , but suppose that the target is  $T = (3USL + LSL)/4$ , so that the specification limit are asymmetric. Then,

suppose that  $\sigma = (USL - LSL)/3$ . It is easy to show that  $C_{pk}^* = C_p^*(1, 0) = 0$  and  $C_{pmk}^* = C_p^*(1, 1) = 0$ , but, under the assumption of normality, the process will give approximately 0.27% non-conforming items. This fact proves that the process capability is severely underestimated.

A second way to modify the true specification limits to get a symmetric interval is found in Kushler and Hurley (1992) and Franklin and Wasserman (1992). Both of the specification limits are shifted by considering a new interval centred on  $T$  whose half-width is given by the mean of the distances  $USL - T$  and  $T - LSL$ , that is  $(USL - LSL)/2 = d/2$ . Notice that the new interval  $(T \pm d/2)$  has the same width of the original one, but it is not necessarily centred on  $M = (USL + LSL)/2$ . It is easy to get:

$$C'_p(u, v) = \frac{d - u|\mu - T|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}. \quad (31)$$

Again, the indices belonging to class (31) can raise misleading judgments about the real capability of the process. Chen and Pearn (2001) report the following example: consider two processes (named, for the sake of simplicity, as A and B) with the same target  $T = (3USL + LSL)/4$  but different means,  $\mu_A = T - USL + LSL$  for process A and  $\mu_B = USL$  for process B. Further, suppose that  $\sigma = (USL - LSL)/6$  for both processes and that normality holds. If  $C'_{pk} = C'_p(1, 0)$  and  $C'_{pmk} = C'_p(1, 1)$  are considered, then both indices will indicate a better capability of process B ( $C'_{pk} = C'_{pmk} = 0$  for process A and  $C'_{pk} = 1.0$ ,  $C'_{pmk} = 0.32$  for process B), even if the latter process is expected to provide approximately 50% non-conforming items against a percent of about 0.135% which characterizes process A.

Boyles (1994) notices that the classical  $C_{pm}$  in (4) can be reinterpreted as the ratio  $1/(3\sqrt{\xi})$ , where  $\xi = E\left\{\frac{(X-T)^2}{(USL-LSL)^2}\right\}$  i.e. the expectation of the loss function  $\mathcal{L}(x) = \frac{(x-T)^2}{(USL-LSL)^2}$ . By considering the piecewise quadratic loss function

$$\mathcal{L}'(x) = \begin{cases} \frac{(x-T)^2}{(T-LSL)^2} & \text{for } x \leq T \\ \frac{(x-T)^2}{(USL-T)^2} & \text{for } x > T, \end{cases} \quad (32)$$

Chen and Pearn (2001) defines the generalized class (33) for asymmetric tolerances:

$$C_{pm}^* = \frac{1}{3} \left( \frac{W_-^2}{(T - LSL)^2} + \frac{W_+^2}{(USL - T)^2} \right)^{-1/2} \quad (33)$$

where  $W_-^2$  and  $W_+^2$  are defined as for (28). When  $X$  has a normal distribution,  $W_-^2 = \sigma^2 h\left(\frac{T-\mu}{\sigma}\right)$  and  $W_+^2 = \sigma^2 h\left(\frac{\mu-T}{\sigma}\right)$  where  $h(x) = (1+x^2)\Phi(x) + x\phi(x)$  and

$\phi(\cdot)$  denotes the pdf of a standard-normal variable. Hence, for a normal process,

$$C_{pm}^* = \frac{1}{3\sqrt{2}} \left\{ \frac{1}{2} \left[ \left( \frac{T - LSL}{\sigma \sqrt{h\left(\frac{T-\mu}{\sigma}\right)}} \right)^{-2} + \left( \frac{USL - T}{\sigma \sqrt{h\left(\frac{\mu-T}{\sigma}\right)}} \right)^{-2} \right] \right\}^{-1/2}. \quad (34)$$

Notice that, up to the multiplicative constant  $\frac{1}{3\sqrt{2}}$ , (34) can be regarded as a mean of the two ratios into round parenthesis, according to the power function  $f(x) = x^{-2}$  (see Hardy *et al.*, 1952). Boyles (1994) proposes to consider also the index

$$C_{pm}^+ = C_{pm}^* \sqrt{\frac{1 + \min\{r^2, 1/r^2\}}{2}} \quad \text{where } r = \frac{T - LSL}{USL - T}. \quad (35)$$

The index  $C_{pm}^+$  is built so that it can take the same value of  $C_{pk}$  in (3) when  $\mu = T$ , independently of the asymmetry of tolerances. This fact can guarantee a closer link to the process yield, as below detailed. The indices  $C_{pm}^*$  and  $C_{pm}^+$  can be related to the index  $C_{jkp}$  in (28), proposed in Johnson *et al.* (1994), which was referred to in Section 2. For a normal process, the index can be re-written as:

$$C_{jkp} = \frac{1}{3\sqrt{2}} \min \left\{ \frac{T - LSL}{\sigma \sqrt{h\left(\frac{T-\mu}{\sigma}\right)}}, \frac{USL - T}{\sigma \sqrt{h\left(\frac{\mu-T}{\sigma}\right)}} \right\}, \quad (36)$$

where the function  $h(\cdot)$  is defined as for (34). By comparing (34) and (36), it can be noticed that, in the Normal case, they are different means of the same ratios. According to the properties of such means, one can then conclude that  $C_{pm}^* \geq C_{jkp}$ . Boyles (1994) evaluates the performance of  $C_{jkp}$  when asymmetric tolerances are considered; to give a closer link to the process yield, a smooth version of  $C_{jkp}$  is also proposed

$$S_{jkp} = \frac{1}{3\sqrt{2}} S \left( \frac{T - LSL}{\sigma \sqrt{h\left(\frac{T-\mu}{\sigma}\right)}}, \frac{USL - T}{\sigma \sqrt{h\left(\frac{\mu-T}{\sigma}\right)}} \right). \quad (37)$$

By looking at the function  $S(\cdot, \cdot)$  defined in (8),  $S_{jkp}$  can be regarded as a mean of the two arguments in (37) according to the function  $f(x) = \Phi(x)$ . This fact implies that, in the normal case,  $S_{jkp} \geq C_{jkp}$ . Unfortunately, a relation between

$S_{jkp}$  and  $C_{pm}^*$  cannot be straightforwardly derived.

Boyles (1994) compares some of the above reported indices by the following argumentation: if a given index is greater or equal to a fixed value  $c$ , then the process should have a proportion of conforming items at least equal to  $2\Phi(3c) - 1$ , which is the yield corresponding to  $S_{pk} = c$  in the normal case (remember the definition (7) given in Section 1). Such a property approximately holds for  $S_{pmk}$  defined in (9) when  $\mu$  is in a neighbourhood of the target, regardless of the position of the latter with respect to the limits. Actually, the same fact holds true even for  $C_{pmk}$  defined in (5), but only when  $c$  is sufficiently large. As regards the indices defined in (34)–(37),  $C_{pm}^* \geq c$  can give a process yield lower than  $2\Phi(3c) - 1$  in the asymmetric case;  $C_{pm}^+ \geq c$  can also give a lower yield, but not as much as for  $C_{pm}^*$ , at least when  $c > 1$ ; finally, the same drawback can be experienced both for  $C_{jkp}$  and  $S_{jkp}$  in the asymmetric case.

The reported comparison by Boyles (1994), along with the above criticism of  $C_p^*(u, v)$  in (30) and of  $C_p'(u, v)$  in (31), reflects the concern that the value taken by a capability index should not contradict a given extent of the process yield. However, a fair process capability analysis should not mask the importance of the target as well. It is easily shown that indices such as  $S_{pmk}$  and  $C_{pmk}$  (like  $C_{pm}^*$ ,  $C_{pm}^+$ ,  $C_{jkp}$  and  $S_{jkp}$ ) do not take their maximum value at  $\mu = T$ , but at different values in the interval between the target and  $M$ . Such a drawback is emphasised in the following example by Pearn and Chen (2001): let  $LSL = 26$ ,  $USL = 58$  and  $T = 50$  and take two processes  $A$  and  $B$  both with standard deviation  $\sigma = 5.33$  and with means  $\mu_A = 49$  and  $\mu_B = 50$ . The above six indices are lower the process  $B$  which is, however, different from  $A$ , on target.

The latter example shows that, due to the asymmetry of tolerances, yield and centering could be conflicting capability criteria. To better understand the origins of such a conflict and to propose suitable corrections, a comparison of some indices of the two classes (6) and (30) can be conducted. Consider first the numerator of  $C_{pk} = C_p(1, 0)$ ,  $d - |\mu - M|$ , that is the distance of the process mean from the closest specification limit. Such a distance is directly related to the yield, as it is half the range of the process values which both are most probable and meet the specification limits. In the numerator of  $C_{pk}^* = C_p^*(1, 0)$ , the same logic is applied after “shifting” one of the specification limits to get a symmetric interval; this fact makes one recover the importance of the target, even if no changes in the numerators of the two indices occur, depending on the position of  $\mu$  respect to  $T$ . For instance, let  $D_l = T - LSL$ ,  $D_u = USL - T$  and suppose that  $D_l > D_u$ , so that the lower specification limit is shifted to  $LSL' = 2T - USL$ . The

numerator of  $C_{pk}^*$  equals the one of  $C_{pk}$  only when  $\mu \geq T$ . In this case, process centering and process yield are not conflicting criteria, in that they both get worse as  $\mu$  moves away from  $T$ . Conversely, when  $\mu < T$ , the numerator of  $C_{pk}^*$  will be generally reduced with respect to the one of  $C_{pk}$ , as an effect of the centering. An interesting case is when  $LSL' < \mu < T$ ; in this case, the capability of the process, as measured by  $C_{pk}^*$ , worsens the more  $\mu$  gets lower than  $T$ , regardless of the yield which could be instead very good (surely better if  $M < \mu < T$ ). The cause of this drawback is that the numerator of  $C_{pk}^*$  gives the same weight to equal deviations of the mean from  $T$ , regardless of how close  $\mu$  is to the (original) specification limits. A good correction could then be to re-weight such deviations if they occur in the side of the interval which was forcedly “shifted”. For instance, if, again,  $D_l < D_u$  and  $LSL' < \mu < T$ , the deviation  $T - \mu$  in the interval  $(LSL', T)$  (whose width is  $d^* = \min\{D_l, D_u\}$ ) must be given the same weight as the one in the interval  $(LSL, T)$  (whose width is  $D_l$ ). The new distance of the mean from the limit  $LSL'$  will then be  $d^* - (T - \mu) \frac{d^*}{D_l}$ . This is the numerator of a new index proposed by Chen and Pearn (2001):

$$C_{pk}'' = \frac{d^* - F^*}{3\sigma}, \tag{38}$$

where  $F^* = \max\left\{(\mu - T) \frac{d^*}{D_u}, (T - \mu) \frac{d^*}{D_l}\right\}$ .

The possible conflict between yield and centering can affect other parts of a capability index. Consider now  $C_{pm} = C_p(0, 1)$  and notice that it has the same denominator as  $C_{pm}^* = C_p^*(0, 1)$ . The asymmetry of tolerances can cause a conflict between yield and the centering in the denominators of both indices. Suppose again that  $D_l > D_u$  (which implies that  $T > M$ ) and consider that, for a given  $\sigma$ ,  $C_{pm}^*$  increases as  $\mu$  gets closer to  $T$ . When  $\mu < M$  or  $\mu > T$ , getting closer to  $T$  implies that the process has both a better centering and a higher yield, which can be measured by the distance of  $\mu$  from  $M$ . Conversely, when  $M < \mu < T$ , an increase of the index will reflect a better centering but not a lower yield. Such a conflict depends, of course, on the asymmetry of the limits and could be solved by giving a different weight to the distance  $(T - \mu)$ , (which has then to be squared). More specifically, one can re-weight  $T - \mu$  in the interval  $(LSL, M)$  as it weights in the interval  $(LSL, T)$ . This means that  $T - \mu$  is to be multiplied by the ratio  $d/D_l$ , so that the following index is obtained:

$$C_{pm}'' = \frac{d^*}{3\sqrt{\sigma^2 + F^2}}, \tag{39}$$

where  $F = \max\left\{(\mu - T) \frac{d}{D_u}, (T - \mu) \frac{d}{D_l}\right\}$ . By following both the logics leading to

(38) and (39), a generalisation of  $C_{pmk}$  in (5) can then be considered as follows:

$$C''_{pmk} = \frac{d^* - F^*}{3 \sqrt{\sigma^2 + F^2}}, \quad (40)$$

where  $F^*$  and  $F$  are defined as above. Actually (38), (39) and (40) belong to a Vännman-like class

$$C''_p(u, v) = \frac{d^* - uF^*}{3 \sqrt{\sigma^2 + vF^2}}. \quad (41)$$

Chen and Pearn (2001) define (41) with the aim of getting an index which can be both respectful of the yield and capable of taking its maximum value for  $\mu = T$ , regardless of the asymmetry of limits. As a consequence, the class (41) is built so that, if two processes  $A$  and  $B$ , sharing the same  $\sigma$ , are such that  $(\mu_A - T)/D_u = (T - \mu_B)/D_l$ ,  $C''_p(u, v)$  will return the same value to both the processes for every  $(u, v)$ . This is a natural requirement as it implies that the index decreases faster as  $\mu$  moves away from the target in the direction of the closer specification limit, which can be considered as the most “dangerous” direction.

The problem of asymmetric tolerances is somewhat related to the one of one-sided tolerances. The quality of some productions is often measured by a quantity which has naturally a one-sided boundary; consider, for instance, the roughness of a surface, which cannot be lower than zero. This chances have been extensively considered by the one-sided versions of the classical indices, starting from the couple (2) referred to in the Introduction up to their many variations. There are, however, other cases where a drift of the mean in a specific direction may be simply considered as more or less “serious” than the one in the opposite direction. In a sense, tolerances are here still asymmetric as the final user may be able to state that the risk to drift in, say, the upper direction is  $k$  times more serious than the risk in the lower direction, so that one can set  $D_l = kD_u$ . One can then consider the class of indices

$$C^U_p(u, v) = \frac{USL - T - uF^*_U}{3 \sqrt{\sigma^2 + vF^{*2}_U}}, \quad (42)$$

where  $F^*_U = \max\{(\mu - T), (T - \mu)\frac{1}{k}\}$ . The class (42), which was proposed by Grau (2009), coincides with another class of indices for the one-sided upper tolerance by Vännman (1998),

$$C^U_{pv}(u, v) = \frac{USL - T - u|\mu - T|}{3 \sqrt{\sigma^2 + v(\mu - T)^2}}, \quad (43)$$

only when  $\mu > T$ . Grau (2009) underlines, however, that the class (43) is not convenient as it considers symmetrically any drift of the mean with respect to

the target, independently of its direction. Indeed, when  $\mu < T$ , the two classes (43) and (42) give different indices, because the latter will consider the minor “seriousness” of the drift of the mean towards the opposite side of the relevant tolerance. It is interesting to notice that (42) belongs indeed to the general class of PCIs for asymmetric tolerances (41) defined by Chen and Pearn (2001). In the above notation, when the one-sided upper index is considered, it is easily shown that  $d^* = D_u$ ,  $F^* = F_U^*$  and  $F = \frac{1}{2}(1+k)F_U^*$ , so that

$$C_p^U(u, v) = \frac{d^* - uF^*}{3 \sqrt{\sigma^2 + v \frac{4}{(1+k)^2} F}} = C_p'' \left( u, \frac{4}{(1+k)^2} \right). \tag{44}$$

The problem of asymmetric tolerances seems to have reached a definite solution with the indices defined by Chen and Pearn (2001). With some minor exceptions, the recent literature is indeed often limited to some marginal developments about the class (41), without really discussing its validity under the usual assumptions. For instance, Grau (2011) has recently investigated the mathematical relationship between the class (41) and the expected percentage of conforming items; such a relationship turns out to be technically rather complicate, compared with the one for other indices, like  $S_{pmk}$  in (9). Another interesting development about the class (41) concerned the chance that the distribution of the process characteristic  $X$  may be non-normal and even asymmetric. This issue has been detailed in section 2 for classical indices, but it deserves further attention here. It is well known that, to obtain a suitable index for non-normal processes,  $\sigma$  can be replaced by  $\frac{1}{6} [x_{0.99865} - x_{0.00135}]$  and  $\mu$  by the median  $x_{0.5}$  (see Clements, 1989 and Pearn and Kotz, 1994). By applying such a logic to (41), the resulted index is

$$C_{Np}''(u, v) = \frac{d^* - uA^*}{3 \sqrt{\frac{1}{36} [x_{0.99865} - x_{0.00135}]^2 + vA^2}}, \tag{45}$$

where  $A^* = \max \left\{ (x_{0.5} - T) \frac{d^*}{D_u}, (T - x_{0.5}) \frac{d^*}{D_l} \right\}$  and  $A = \max \left\{ (x_{0.5} - T) \frac{d}{D_u}, (T - x_{0.5}) \frac{d}{D_l} \right\}$  (see Pearn *et al.* (1999) and Pearn *et al.* (2005)). Clearly, the most relevant members of the class (45) are obtained by letting  $u$  and  $v$  take the values 0 and 1. Moreover, such members can be all written in terms of the index obtained for  $u = v = 0$ , i.e.  $C_{Np}'' = C_{Np}''(0, 0)$  :

$$C_{Npk}'' = C_{Np}''(1, 0) = (1 - \alpha) C_{Np}'' \tag{46}$$

$$C_{Npm}'' = C_{Np}''(0, 1) = (1 + \delta^2)^{-1/2} C_{Np}'' \tag{47}$$

$$C_{Npmk}'' = C_{Np}''(1, 1) = (1 - \alpha)(1 + \delta^2)^{-1/2} C_{Np}'' \tag{48}$$

where  $\alpha = A^*/d^*$  and  $\delta = 6A/[x_{0.99865} - x_{0.00135}]$ . Grau (2010) shows that a non-normal distribution is not necessarily symmetric; to take into account the possibilities that  $X$  has a right- or a left-skewed distribution, an alternative to  $C''_{Np}$  is then proposed,

$$C^{\#}_{Np} = \min \left\{ \frac{D_u}{x_{0.99865} - x_{0.5}}, \frac{D_l}{x_{0.5} - x_{0.00135}} \right\}, \quad (49)$$

so that some new relevant indices can be defined by substituting  $C^{\#}_{Np}$  for  $C''_{Np}$  in (46)–(48). Such indices reduce to the ones of the class (45) when  $X$  has a symmetric distribution; if, in addition,  $X$  follows the normal distribution, they reduce to indices of the class (41). Furthermore, if the tolerances are symmetric respect to the target and the process is normal, the above indices reduce to the ones of the Vännman class in (6).

Even the recent indices by Grau (2010) can be regarded as a further development about the class (45), whose paradigm, however, is not essentially discussed. A substantial change of view, in this sense, could derive from other papers making use of the logic of fuzzy sets, even if the research in this direction seems to be still limited. Parchami and Mashinchi (2010) emphasise that, since specification limits are often non-precise numbers, they could be better expressed in fuzzy terms. After defining an upper and a lower fuzzy specification limit, their distance replaces the crisp number  $2d = USL - LSL$  in the re-definition of classical PCIs. There are, however, some limitations to the fuzzy nature of the new indices by Parchami and Mashinchi (2010), as the mid-point  $M$  of the specification interval, as long as the target value  $T$ , are still considered as crisp numbers. As a consequence, this research, though quite promising in the way it was directed, needs further improvements (see also Yongting, 1996; Lee, 2001; Tsai and Chen, 2006 and Parchami and Mashinchi, 2009).

#### 4. CONCLUSIONS AND POSSIBLE DEVELOPMENTS

This paper focused on the recent and past developments in the theory of univariate indices of capability when the classical assumptions regarding the process and the specification limits are not met. A strong assumption upon which classical indices are based is normality. Section 2 reviewed some recent contributions in which the process characteristic follows a different, possibly unspecified, distribution. Section 3 faced instead the problem of asymmetric tolerances and the consequent need to define more flexible capability indices, able to mediate between the possibly conflicting criteria of yield and centering.

Among the issues investigated in this paper, an efficient estimation of PCIs has been as well dealt with. Inference is, indeed, a crucial aspect when measuring the capability of a process, especially if confidence intervals and hypothesis testing are to be conducted. If, in fact, finding an unbiased estimator is already a non-trivial task, deriving its exact distribution frequently reveals impracticable. On the other hand, turning to asymptotic theory is not a possible solution for many indices. These issues result to be more complicate matters if the new-generation indices for non-normal processes and asymmetric tolerances, reviewed in Sections 2 and 3, are taken into consideration. A partial remedy could come from simulations, which are often the basic tool used by many authors both to assess the properties of estimators and to compare their performances. An appropriate software and a sufficient processing capability are then needed both for theoretical and practical purposes.

Furthermore, the classical literature assumes that the sample is composed by identically and independently distributed random variables. Unfortunately this condition is hard to find in some industrial processes, like in the tool wear case or in the chemical industry. Some techniques to remove serial correlation before estimating the capability of a process are known. Montgomery (1985) proposed to fit an AR(1) model to autocorrelated data. Modeling trend was also suggested by Alwan and Roberts (1988) who also proposed to use residuals for process monitoring. Other approaches are based on suitable modification of a PCI due to serial correlation. When systematic assignable causes are present and tolerated, the variability of the process can be divided into two parts: one due to random causes ( $\sigma_r^2$ ) and the other due to assignable causes ( $\sigma_a^2$ ), that is  $\sigma^2 = \sigma_r^2 + \sigma_a^2$ . Spiring (1991), thinking of a dynamic process in constant change, proposed a modification of the  $C_{pm}$  index in (4) by estimating the variance of the process just with  $\sigma_{r_t}^2$ , that is the variability due to random causes only at time  $t$ . Wallgren (1996) studied the effect on  $C_{pm}$  when consecutive measurements are represented as observations from a Markov process in discrete time and presented a modification of the  $C_{pm}$  index.

Many other authors (Zhang *et al.*, 1990; Shore, 1997; Zhang, 1998; Scholz and Vagel, 1998 and Noorossana, 2002) investigated the properties of the classical indices in case of autocorrelated data. It seems, however, that the problem of serially correlated data has received a relatively small attention in recent literature. The authors are convinced that this gap should be filled and some new efforts should be paid to examine in-depth this topic.

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