STOCHASTIC VOLATILITY MODELS IN INVESTMENT CHOICES

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Abstract

The goal of this article is to show how stochastic volatility models can support investment choices made by traders and quantitative fund managers in their activities. The study of the most important features empirically observed in many financial markets will justify the use of stochastic volatility models. In particular, we will concentrate on Heston model (the most used by practitioners among stochastic volatility models) discussing its major advantages and limitations. In the last part of the article is shown the use of Heston model in the analysis of certain derivatives instruments where volatility plays an important role.

Keywords: Stochastic volatility; Heston model; implied volatility; variance swap; volatility surface.

1. INTRODUCTION

Volatility is a key factor in many financial applications: pricing of financial instruments, asset allocation, risk management.

The accurate estimation of such a variable is fundamental for the pricing of derivatives instruments; pricing models, in fact, require the estimation of the underlying expected volatility as input variable.

The word “volatility” is very used in financial terminology with different meanings: “realized volatility” (or historical volatility), “implied volatility”, “normalized volatility”, “local volatility”.

The realized volatility of a financial asset is the annualized standard deviation of its return (continuously compounded) during a period (usually one year).

The implied volatility refers to the volatility parameter that has to be put in Black-Scholes formula to get the market price of a specific option. We will further examine this aspect in the next sections.

* This article represents only the personal opinions of the author and not necessarily those of Fondiaria-SAI, its subsidiaries or affiliates.
The “normalized volatility” is the product between implied volatility and underlying price. This quantity is very used in fixed income markets.

“Local volatility” is the (state-dependent) coefficient of the unique diffusion process consistent with the prices of a given set of listed options [Derman E., Kani I. (1994); Dupire B. (1994)].

During the last decades, financial institutions have understood the importance of volatility and discovered that decisions made at every firm level are more and more influenced by this variable. The need for theories of volatility became evident and increasing resources have been devoted for this purpose.

On the other hand, academic researchers became more aware on the importance of volatility and an impressive literature flourished.

The goals of this article are twofold. The first is making a brief excursus of existing models that could reproduce, or in the better cases, could explain what is empirically observed on financial markets; the second is to show how Heston model could be used not only for pricing derivatives instruments, but also for making advanced risk analysis. We will show how variance swaps could be priced and studied using this model.

2. OPTION MARKETS

On the markets of listed options can be found the prices of “plain vanilla” options (call and put options) for different strikes ($K$) and maturities ($T$) determined by supply and demand conditions.

There are not listed options for all financial assets. For example, this is the case of many shares. Moreover, the number of listed options is different for each security. For example, there are more listed options on equity indexes than options on single stocks.

Another important factor is liquidity: traded volumes can differ substantially for each contract; the consequence is wider bid-offer spreads for less traded options.

The price determination is not dictated by any theoretical model; nevertheless market prices are “converted” in implied volatilities, inverting Black-Scholes formulas.

In this way, for each pair $(K,T)$ corresponds an implied volatility parameter $\sigma^{(K,T)}_{imp}$.

Conceptually this is equivalent to ask which volatility parameter characterizes the diffusion process of the underlying consistent with the market price, assuming the validity of Black-Scholes (BS henceforth) model.

This market practice could induce to believe that market operators assume the validity of BS model: however this is not the case.

Implied volatility observed on financial markets are different for each pairs
of \((K,T)\) and this is clearly in contrast with the constant volatility assumption made in BS model.

So, although market operators use BS formulas, nobody believes that BS model is a consistent representation of reality. Quoting N. Taleb: “Implied volatility is the wrong number to put in the wrong model that gives the right market price”.

In the chart 1 is shown the volatility surface of equity index DJ Eurostoxx50. The plot of the implied volatility at a given maturity for different strikes can exhibit the so-called volatility skew or smile.

In the case of skew, implied volatility is decreasing for higher strikes; in the case of smiles, implied volatility increases to the extent that strikes are far from the at-the-money value.

In the chart 2 is plotted an example of implied volatility shapes.

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**Chart 1: Volatility surface of Dow Jones Eurostoxx 50 equity index.**

**Chart 2: Skew and smile as observed on equity and FX option market respectively.**
3. EXPLANATIONS FOR SKEW AND SMILE

The presence of skew and smile can be explained in two ways. The first is theoretical and concerns the statistical properties of the underlying distribution; the second examines the structure of option markets.

The prices of options expiring at time $T$ contain information about the implied risk-neutral probability of the underlying at expiry. This distribution can be determined from a set of option prices with continuous strikes and maturity at time $T$. In fact, assuming that the evolution of the underlying $S$ is governed by a diffusion process, the implied risk-neutral probability density (PD) can be derived:

$$
\bar{C}(K, T) = \int_{K}^{\infty} \varphi(S_r, T)(S_r - K) dS_r
$$

(1)

$$
\frac{\partial \bar{C}}{\partial K} = -\int_{K}^{\infty} \varphi(S_r, T) \vartheta(S_r - K) dS_r
$$

(2)

$$
\frac{\partial^2 \bar{C}}{\partial K^2} = \int_{K}^{\infty} \varphi(S_r, T) \delta(S_r - K) dS_r = \varphi(K, T)
$$

(3)

where: $\bar{C}(K, T)$ is the undiscounted price of a call option with maturity $T$ and strike $K$, $\varphi(S_r, T)$ indicates the (risk-neutral) implied PD of the underlying $S$, $\vartheta(\bullet)$ is the Heaviside function and $\delta(\bullet)$ is the Dirac delta function.

From (1)-(3) can be seen that skew and smiles are intimately linked with statistical properties of risk-neutral PD. In particular, skew in implied volatility results from skewness in log-return distribution; smiles are originated, instead, from kurtosis.

Economic arguments, familiar to market operators, can help to better understand the origin of skew and smile found in different market compartments.

Implied volatility of equity options exhibits a skew shape. This means that out-of-the-money (OTM) put and in-the-money (ITM) call are relatively expensive. The aversion of market operators to extremely negative events creates demand for OTM put, pushing up the prices (and consequently the implied volatility) for these contracts.

Moreover, a popular portfolio strategy consists in buying a stock and selling OTM calls (written on that stock). This creates a strong supply for these contracts making them cheap (the implied volatility is low).

On the fixed-income option markets the common shape of implied volatility
is smile. As in the previous case, the risk aptitude of market operators is crucial. In fact, on this market can be found two kind of operators. The first kind are agents whose liabilities are linked to floating rates and, to hedge against a heavy raise in interest rate, buy OTM caps (equivalents to call contracts on interest rates). On the other hands there are operators whose assets are linked to floating rates and, to hedge against heavy falls in interest rate, buy floors (equivalents to put contracts on interest rates).

A typical shape of implied volatility on FX option market is smile.

Exchange rate is a conversion ratio between two currencies. On this market there is a symmetric demand for protection against extreme events. This means that there are operators seeking protection on a currency and others requesting protection on the other. Let’s consider, for instance, the market for the options written on the euro/dollar exchange rate. Americans will seek protection against euro appreciation and/or dollar depreciation while europeans will request protection against euro depreciation and/or dollar appreciation.

On commodity option markets we will observe a positive skew (the implied volatility increases as strikes become higher) because the negative event is the increase of commodity prices.

4. SKEW AND SMILE MODELS

Considering skew and smile is very important for pricing exotic options, the contracts not belonging to plain vanilla options.

Pricing models used for the valuation of exotic options make assumptions about strategies and hedging instrument and, often, they include plain vanilla options. Is then important that a pricing model reproduces, or in better cases, explains what is observed on option markets.

Today, OTC (over-the-counter) options have very complex payoffs. Accurate pricing models are necessary both for sellers (in order to avoid serious mispricing) and for buyers (to assess and evaluate risks).

During the last two decades, have been studied new models that met (often partially) the needs of operators.

Hereafter will be described the principal classes of models.

Local volatility models [Derman E., Kani I. (1994); Dupire B. (1994)]. The basic idea at the base of this class of models is that a given set of option prices can be replicated assuming that the underlying evolution can be described by a diffusion process whose volatility is a function locally dependent by the underlying price and the time-to-maturity. Implied volatility of a given option struck at \( K \) can be thought as the mean of all local volatilities along the most probable paths of the underlying
till maturity date. These models have two main drawbacks: instability of calibrated parameters and unrealistic evolution of forward volatility.

Stochastic volatility models [Heston S.L. (1993); Hull J.C., White A. (1987); Scott L. (1987); Stein E., Stein J. (1991)]. In stochastic volatility models volatility is itself a stochastic variable (not just a parameter as in BS model or a deterministic function as in local volatility models). This macroclass will be examined extensively in the next sections.

Stochastic volatility models with jumps [Bates D. (1996)]. This class of model extends the previous family, allowing jumps in the underlying process. The main advantage over the simple stochastic volatility models is a better fit of short-maturity implied volatility; the main disadvantages can be found in calibration phase, especially in the estimation of jump parameters.

5. FEATURES OF VOLATILITY: AN EMPIRICAL ANALYSIS

Studies made on different markets [Cont R. (2005)] show some common features of volatility:

a) High volatility. Some empirical studies point out that realized volatility is often not justified by the variability of macroeconomics variables. In particular, huge variations in returns do not always correspond with the arrival of new information.
b) Fat tails. Return distribution is usually characterized by fat tails (excess-kurtosis).
c) Volatility clustering. As noted by Mandelbrot, “big variations in returns tend to cluster in some periods”. This results in the alternation of periods with high volatility and period with low volatility.
d) Mean Reversion. Volatility tends to decrease (increase) after reaching high (low) levels. Heuristic arguments could justify this feature: for example, let’s consider the stock volatility in the next century. If volatility were not mean reverting, its value could probably lie out of the range of 1%-100% in the next 100 years. This is in contrast with the common sense.
e) Correlation between exchanged volume and volatility. Empirically can be observed a positive correlation between the trading volume and volatility.

6. STOCHASTIC VOLATILITY MODELS

In stochastic volatility models, underlying and volatility are separately modeled; each variable is describes by a different model.

Unlike local volatility models, where volatility is a deterministic function of the underlying and time, in stochastic volatility models volatility is itself a stochastic state variable.
Some of the existing models can be summarized in this way [Fouque J.P., Papanicolaou P., Sircar K. (2000)];

\[
dS(t) = \mu S(t)dt + \sigma(t)S(t)dZ_1
\]

\[
\sigma(t) = f(Y(t))
\]

and \( Y(t) \) modeled with:

a) log-normal process: \( dY(t) = c_1 Y(t)dt + c_2 Y(t)dZ_2 \);

b) Ornstein-Uhlenbeck process: \( dY(t) = \alpha \left( \bar{Y} - Y(t) \right) dt + \beta dZ_2 \);

c) Feller process, also known in finance as Cox-Ingersoll-Ross process [Cox J.C., Ross S. (1976)] (CIR): \( dY(t) = \alpha \left( \bar{Y} - Y(t) \right) dt + \sqrt{Y(t)}dZ_2 \), where \( dZ_1 \) e \( dZ_2 \) are two Wiener processes.

Table 1 shows some important features of the most popular models.

**Table 1: Summary of most important stochastic volatility models**

<table>
<thead>
<tr>
<th>Author(s) (Year)</th>
<th>Correlation ( \langle dZ_1, dZ_2 \rangle )</th>
<th>( f(Y(t)) )</th>
<th>Model for ( Y(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hull-White (1987)</td>
<td>( \langle dZ_1, dZ_2 \rangle = 0 )</td>
<td>( \sqrt{Y(t)} )</td>
<td>Log-normal</td>
</tr>
<tr>
<td>Scott (1987)</td>
<td>( \langle dZ_1, dZ_2 \rangle = 0 )</td>
<td>( e^{Y(t)} )</td>
<td>Ornstein-Uhlenbeck</td>
</tr>
<tr>
<td>Stein-Stein (1991)</td>
<td>( \langle dZ_1, dZ_2 \rangle = 0 )</td>
<td>(</td>
<td>Y(t)</td>
</tr>
<tr>
<td>Ball-Roma (1994)</td>
<td>( \langle dZ_1, dZ_2 \rangle = 0 )</td>
<td>( \sqrt{Y(t)} )</td>
<td>CIR</td>
</tr>
<tr>
<td>Heston (1993)</td>
<td>( \langle dZ_1, dZ_2 \rangle \neq 0 )</td>
<td>( \sqrt{Y(t)} )</td>
<td>CIR</td>
</tr>
</tbody>
</table>

Models represented in table 1 has closed-form (or almost closed-form) solutions for the pricing of plain vanilla derivatives. This is particularly important for model calibration.

7. **HESTON MODEL**

Among stochastic volatility models, Heston model is one of the most (if not the most) important.

This model is relatively flexible and, with little adjustments, is used for trading and structuring purposed by most financial institutions.

In Heston framework underlying and volatility are separately modeled. The latter, unlike the BS model where is just a parameter, is a state variable.

This model uses two SDEs (stochastic differential equations) that, under the real probability measure \( P \), assume the following form:
\[ dS(t) = \mu S(t)dt + \sqrt{V(t)}S(t)dZ_1 \] (4)

\[ dV(t) = \kappa(\theta - V(t))dt + \sigma\sqrt{V(t)}dZ_2 \] (5)

\[ \langle dZ_1, dZ_2 \rangle = \rho dt \] (6)

The dynamics of the underlying \( S \) is described by a geometrical Brownian motion, while the variance \( V \) by a Feller process (or CIR process).

Variance will be strictly positive if the following condition holds:

\[ 2\kappa\theta > \sigma^2 \] (7)

All parameters appearing in (4)-(6) bear a specific financial meaning:
\( \mu \) = Expected growth rate of \( S \);
\( \theta \) = Long-term variance;
\( \kappa \) = Speed of mean reversion;
\( \sigma \) = Variance volatility (also known as vol-of-vol).

Under the risk-neutral measure (4)-(6) assume the following form:

\[ dS(t) = (r - q)S(t)dt + \sqrt{V(t)}S(t)dZ_1 \] (8)

\[ dV(t) = \kappa^* \left( \theta^* - V(t) \right)dt + \sigma\sqrt{V(t)}dZ_2 \] (9)

\[ \langle dZ_1, dZ_2 \rangle = \rho dt \] (10)

with \( \kappa^* = \kappa + \lambda \) and \( \theta^* = \frac{\kappa\theta}{\kappa + \lambda} \), where \( \lambda \) is the market volatility risk, \( r \) the riskless rate and \( q \) the dividend yield.

Log-return PD (and consequently the underlying distribution) at maturity depends by model parameters.

In particular, high values for vol-of-vol parameter generate kurtosis; correlation coefficient controls the skewness.

In Heston article [Heston S.L. (1993)] can be found the derivation of the closed-form formula for the valuation of european call options.

If the dynamics of the underlying \( S \) and volatility \( V \) are those described in (8)-(10), then the price of a european call is given by:

\[ C(S_0, v, K, T) = S_0 e^{-qT} P_1 - Ke^{-rT} P_2 \] (11)
with

\[ P_j \left( x, v, \ln[K], T \right) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left( \frac{e^{-i\phi \ln(K)}}{i\phi} \right) f_j \left( x, v, T, \phi \right) d\phi \]

\[ x = \ln(S_0 e^{-qT}) \]

\[ f_j \left( x, v, T, \phi \right) = e^{C(T, \phi) + D(T, \phi) + i\phi x} \]

\[ C(T, \phi) = r i \phi T + \frac{a}{\sigma^2} \left( b_j - \rho \sigma i \phi + d \right) T - 2 \ln \left( \frac{1 - ge^{dT}}{1 - g} \right) \]

\[ D(T, \phi) = \frac{b_j - \rho \sigma i \phi + d}{\sigma^2} \left( \frac{1 - e^{dT}}{1 - ge^{dT}} \right) \]

\[ g = \frac{b_j - \rho \sigma i \phi + d}{b_j - \rho \sigma i \phi - d}, \quad d = \sqrt{\left( \rho \sigma i \phi - b_j \right)^2 - \sigma^2 \left( 2u_\phi i - \phi^2 \right)} \]

\[ u_1 = 0.5, \quad u_2 = -0.5, \quad a = \kappa \theta, \quad b_1 = \kappa + \lambda - \rho \sigma, \quad b_2 = \kappa + \lambda. \]

Dragulesku et al. [Dragulesku A., Yakovenko V. (2002)] derived the log-return distribution of the underlying \( S \) in Heston model.

If the dynamics of the underlying \( S \) and volatility \( V \) are those described in (8)-(10), then the log-return PD \( x_T \), at maturity \( T \), is given by the following:

\[ P_T \left( x \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x + F_T(\xi)} d\xi \quad (12) \]

\[ x_T = \ln \left( \frac{S_T}{S_0} \right) - \mu T \]

\[ F_T = \frac{\kappa \theta}{\sigma^2} \gamma T - \frac{2\kappa \theta}{\sigma^2} \ln \left( \cosh \left( \frac{\Omega T}{2} \right) + \frac{\Omega^2 - \gamma^2 + 2\kappa \gamma}{2\kappa \theta} \sinh \left( \frac{\Omega T}{2} \right) \right) \]

\[ \gamma = \kappa + i\rho \sigma \xi \]

In [Dragulesku A., Yakovenko V. (2002)] there is an empirical analysis of Heston model. Authors, based upon time series (1981-2001) of the Dow-Jones 30 equity index, conclude that Heston model fits well empirical data.
This model has some advantages and limitations.

Major advantages are:

a) Parsimony. There are few parameters to describe the whole market;
b) Financial meaning of parameters. All parameters have a precise financial meaning. “What if” analysis, very useful for investment choices, can be readily implemented by changing some parameters;
c) Availability of (quasi) closed-form formulas for the pricing of plain vanilla options (very important in the calibration stage). Numerical integrals appearing in (11) can be readily solved by means of numerical quadrature schemes;
d) Frequency of recalibration. It is not necessary to recalibrate frequently the model parameters (unlike local volatility models);

Major limitations are:

a) Inaccurate fits for short-dated options (usually, options with 3/6 months expiry);
b) Multiple set of calibration parameters. Usually, numerical minimization algorithms provide different local solutions (depending on the inputted starting values for the search of minimum);
c) Difficulties in implementing Monte Carlo simulations;
d) Difficulties in extending model to multiple underlyings.

Often, in real life, is necessary to evaluate derivatives depending on multiple underlyings.

In principle, extending Heston models in multiple dimensions is immediate. In the case of two underlyings (not dividend-paying) the model will be:

\[
\begin{align*}
    dS_1(t) &= rS_1(t)dt + \sqrt{V_1(t)}S_1(t)dZ_1 \\
    dV_1(t) &= \kappa_1^*(\theta_1^* - V_1(t))dt + \sigma_1\sqrt{V_1(t)}dZ_1 \\
    dS_2(t) &= rS_2(t)dt + \sqrt{V_2(t)}S_2(t)dZ_2 \\
    dV_2(t) &= \kappa_2^*(\theta_2^* - V_2(t))dt + \sigma_2\sqrt{V_2(t)}dZ_2
\end{align*}
\]

with \( \langle dZ_i, dZ_j \rangle \neq 0 \) for \( i, j = 1, \ldots, 4 \).

As it can be seen from (13)-(16) the number of parameters grows very quickly as the number of underlyings increases, mainly due to correlations among different processes.
8. SIMULATING HESTON MODEL

In many cases, there are not available closed-form formulas for the pricing of exotic derivatives. These contracts have to be priced with the help of numerical methods. European options with a strong path-dependency are usually valuated with Monte Carlo simulations.

To do this, one has to simulate SDEs (8)-(10).

The transition function of the underlying $S$ between the time $u$ and the following period $t$ can be written as:

$$S(t) = S(u) e^{\left[ \frac{(r-q)(t-u)}{2} V(s) ds + \rho \int_{u}^{t} \sqrt{V(s)} dZ_1(s) + \int_{u}^{t} \sqrt{V(s)} dZ_2(s) \right]}$$

(17)

with $\langle dZ_1, dZ_2 \rangle = 0$

Broadie and Kaya showed how to exactly simulate (17); however their method is rather slow and complex (it is necessary to numerically evaluate Bessel function integrals and there is a huge amount of sampling from non-central chi squared distribution).

Practitioners prefer to simulate discretized versions of (8)-(10) with short time steps (short-stepped Monte Carlo).

A discretization method is the Euler scheme. In this case (8)-(10) become:

$$S(t) = S(t-1) + (r-q) S(t-1) \Delta t + \sqrt{V(t-1)} S(t-1) \left[ \rho \Delta Z_1(t) + \sqrt{1-\rho^2} \Delta Z_2(t) \right]$$

(18)

$$V(t) = V(t-1) + \kappa^* \left( \theta^* - V(t-1) \right) \Delta t + \sqrt{V(t-1)} \sigma \Delta Z_1(t)$$

(19)

$$\langle \Delta Z_1, \Delta Z_2 \rangle = 0$$

(20)

The presented scheme has some drawbacks.
When simulating (19), because of discretization errors, the variance $V$ can go negative.
This problem can be circumvented with a couple of tricks:
a) Reflecting condition. In this case $V(t) = \left| V(t) \right|$;
b) Absorbing condition. We impose $V(t) = \max \left( V(t), 0 \right)$.
Choosing one of the two conditions is generally ininfluent.
9. VARIANCE SWAPS

Variance swap is a forward contract where a party (A) pays to a counterparty (B) the realized variance of a financial asset $S$ and receives a predetermined amount; net flows are settled at expiry.

Financial flows for the counterparty (B) will be given by:

$$N \left( \sigma^2_R(S) - K_{\text{var}_\text{swp}} \right)$$

where:

- $N$ is the notional amount;
- $K_{\text{var}_\text{swp}}$ is the variance strike,

$$\sigma^2_R(S) = \sqrt{ \frac{1}{m-1} \sum_{i=1}^{m} \left( \ln \frac{S_i}{S_{i-1}} \right)^2 } \sqrt{252}$$

$m$ is the number of working days between the starting date and the expiration date.

Hence, the counterparty (B) will get (pay) $N$ euro for every point of realized variance exceeding (below) the strike.

Although variance swaps are considered OTC contracts, they are very liquid instruments. Major market players show their quotations (bid/offer) on Bloomberg pages. Market operators, can decide whether to buy or sell variance. Generally, under normal market conditions, bid-offer spread is less than half vega point.

Traders can use variance swaps for hedging some risks resulting from the selling of plain vanilla options.

Option’s (or option portfolio’s) daily P&L (Profit and Loss) is given by:

$$\text{Total P&L} = \Delta \Pi + \Pi_{\Delta S} + \Pi_{\Delta t} + \Pi_{\Delta S^2} + \Pi_{\text{Other}}.$$ (23)

Let’s suppose that volatility, rates and dividends don’t change, and Other is negligible. Expression (23), under the above mentioned hypotheses, can be written as:

$$\text{Total P&L} = \Delta \Pi + \Pi_{\Delta S} + \Pi_{\Delta t} + \Pi_{\Delta S^2}.$$ (24)

Recalling that P&L is a variation, applying Taylor’s theorem we get:

$$\text{Total P&L} = \Delta \Pi = \frac{\partial \Pi}{\partial S} \Delta S + \frac{\partial \Pi}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 \Pi}{\partial S^2} \Delta S^2 + \ldots$$ (25)
If trader follows a delta hedging strategy, expression (25) will become:

$$\text{Total P&L} = \Theta \Delta t + \frac{1}{2} \Gamma \Delta S^2$$  \hspace{1cm} (26)

In BS model, theta and gamma are interrelated:

$$\Theta \approx -\frac{1}{2} \Gamma S^2 \sigma_{imp}^2$$  \hspace{1cm} (27)

After substituting (27) in (26) we’ll get:

$$\text{Total P&L} \approx \frac{1}{2} \Gamma \left[ (\Delta S)^2 - \sigma_{imp}^2 S^2 (\Delta t) \right] = \frac{1}{2} \Gamma S^2 \left[ \frac{(\Delta S)^2}{S^2} - \sigma_{imp}^2 \Delta t \right]$$  \hspace{1cm} (28)

By means of expression (28) the total P&L of a delta neutral option portfolio can be decomposed into the difference between implied and realized volatility scaled by gamma.

Gamma is then a measure of P&L’s variability. The more gamma is high, the more volatile P&L will be. This demonstrates, and partially justifies, a typical behavior observed on financial markets: traders tend to overcharge option premium with high gamma.

Summing up all total daily P&L (28) yields Final P&L:

$$\text{Final P&L} \approx \frac{1}{2} \sum_{i=1}^{m} \Gamma_i S_i^2 \left[ \frac{S_i - S_{i-1}}{S_{i-1}} \right]^2 - \sigma_{imp \ i}^2 (\Delta t)$$  \hspace{1cm} (29)

Final payoff of a variance swap is given by:

$$\text{VarSwapPayoff} = 252 \frac{1}{m-1} \sum_{i=1}^{m} \left( \ln \frac{S_i}{S_{i-1}} \right)^2 - \text{Strike}^2$$  \hspace{1cm} (30)

Then, expression (30) is different from (29) for the scaling factor of daily variance.
10. PRICING ED HEDGING VARIANCE SWAPS UNDER DIFFUSION MODELS

As shown in (1)-(3) risk-neutral PD of underlying at maturity can be derived differentiating twice the undiscounted call prices with respect to strike. Similar arguments can be used to show that put prices can be exploited to obtain the above mentioned PD.

\[ \varphi(K,T) = \frac{\partial^2 \bar{C}}{\partial K^2} = \frac{\partial^2 \bar{P}}{\partial K^2} \] (31)

Let’s suppose we want to price a derivative contract whose payoff is \( g(S_T) \), with \( g() \) a twice differentiable function.

The value of this derivative will be:

\[ E \left[ g\left(S_T\right) \right] = \int_0^\infty \varphi(K,T) g(K) dK = \int_0^F \frac{\partial^2 \bar{P}}{\partial K^2} g(K) dK + \int_F^\infty \frac{\partial^2 \bar{C}}{\partial K^2} g(K) dK \] (32)

where \( F \) is the forward price of underlying \( S \) at maturity.

Integrating twice by parts and recalling (from the put-call parity) that if \( K=F \) then the value of a put expiring at time \( T \) is equal to the corresponding call value, then:

\[ E \left[ g\left(S_T\right) \right] = \left. \frac{\partial P}{\partial K} \right|_0^\infty - \left. \frac{\partial P}{\partial K} \right|_0^F g'(K) dK + \left. \frac{\partial P}{\partial K} \right|_F^\infty \frac{\partial C}{\partial K} g'(K) dK = \]

\[ = g(F) - \int_0^F \frac{\partial P}{\partial K} g'(K) dK - \int_F^\infty \frac{\partial C}{\partial K} g'(K) dK \]

\[ = g(F) - \frac{\partial P(F)}{\partial K} g'(K) \Big|_0^F + \int_0^F \frac{\partial P(K)}{\partial K} g''(K) dK - \frac{\partial C}{\partial K} g'(K) \Big|_F^\infty + \int_F^\infty \frac{\partial C(K)}{\partial K} g''(K) dK = \]

\[ = g(F) + \int_0^F \frac{\partial P(K)}{\partial K} g''(K) dK + \int_F^\infty \frac{\partial C(K)}{\partial K} g''(K) dK \] (33)

From (33) can be seen that a derivative security whose payoff is twice differentiable can be replicated by a strip of plain vanilla options. The only assumption made to derive this result is the diffusive nature of the process for \( S \).

For example, applying expression (33) to a log-contract (derivative security whose payoff at expiry is \( \log(S_T/F) \)) yields:
\[
E \left[ \log \left( \frac{S_T}{F} \right) \right] = \int_0^F P(K) g''(K) dK + \int_F^\infty C(K) g''(K) dK = \\
= -\int_0^F P(K) \frac{1}{K^2} dK - \int_F^\infty C(K) \frac{1}{K^2} dK
\]

(34)

Log-contract and variance swaps are linked.

Setting interest rate and dividends to zero (so, forward price is equivalent to spot price), applying Ito's lemma at the diffusion process we’ll have:

\[
d \log \left( S_t \right) = \frac{dS_t}{S_t} - \frac{1}{2} \sigma_t^2 dt
\]

(35)

Then:

\[
VAR = \frac{1}{T} \int_0^T \sigma_t^2 dt = \frac{2}{T} \int_0^T \frac{dS_t}{S_t} dt - \frac{2}{T} \log \left( \frac{S_T}{S_0} \right)
\]

(36)

Expression (36) shows that realized variance of a financial asset described by a diffusion model can be replicated in the following way:

a) Sell \(\frac{2}{T}\) log-contracts (that can be replicated as shown in (34));

b) Implement a simple hedging strategy consisting in keeping a constant dollar amount of the underlying (so, it will be necessary to sell/buy the underlying when its price goes up/down).

11. PRICING VARIANCE SWAPS IN HESTON MODEL

Variance swap valuation has to be made, as well as other derivative securities, by taking the expectation (under the risk-neutral probability) of the discounted payoff:

\[
P_{\text{var, swap}} = E \left\{ e^{-rT} \left( \sigma^2_r(S) - K_{\text{var, swap}} \right) \right\}
\]

(37)

As can be seen from expression (37), under the hypothesis of constant interest rates, the only unknown quantity \(E \left\{ \sigma^2_r(S) \right\}\) is.

To simplify calculations, realized variance can be approximated in the following way [Brockhaus O., Long D. (2000)]:
\[
\sigma^2_r(S) \approx \frac{1}{T} \int_0^T V(s) ds
\]  
(38)

Since in Heston model variance is a stochastic variable whose expected value is:

\[
E(V) = e^{-\kappa t} (V_0 - \theta^*) + \theta^*
\]  
(39)

Then:

\[
E\left\{ \sigma^2_r(S) \right\} = E\left\{ \frac{1}{T} \int_0^T V(s) ds \right\} = \frac{1}{T} \int_0^T EV(s) ds =
\]

\[
= \frac{1}{T} \int_0^T e^{-\kappa s} (V_0 - \theta^*) + \theta^* ds = \frac{1}{T} \left[ e^{-\kappa*T} \left( T \theta^* - V_0 \right) + \frac{V_0 - \theta^*}{\kappa^*} + T \theta^* \right] =
\]

\[
= \frac{1 - e^{-\kappa T}}{\kappa^* T} (V_0 - \theta^*) + \theta^*
\]  
(40)

After substituting expression (40) in (37) we’ll get:

\[
P_{\text{var}_\text{swap}} = e^{-rT} \left[ \frac{1 - e^{-\kappa T}}{\kappa^* T} \left( V_0 - \theta^* \right) + \theta^* - K_{\text{var}_\text{swap}} \right]
\]  
(41)

At inception, the fair strike will be given by the following:

\[
K_{\text{var}_\text{swap}}^0 = \frac{1 - e^{-\kappa T}}{\kappa^* T} \left( V_0 - \theta^* \right) + \theta^*
\]  
(42)

12. VARIANCE SWAP ANALYSIS IN HESTON MODEL

Pricing liquid instruments (like variance swaps) could be seen as a pure theoretical exercise, but this is not the case. Valuation is, in fact, the first step for interesting analysis. Typically, an investor would like to detect the key drivers for the performance of a given investment.

In the case of variance swaps, would be interesting to analyze its sensitivity to movements in spot or in long-term volatility.

“What if” analysis can be simply implemented using models whose parameters have a precise financial meaning; “black-blox” models, with obscure parameters, are not suitable for this purposes.
We will analyze a variance swap written on DJ Eurostoxx 50 Index, starting on 05/11/2006 and expiring on 06/15/2007. Model parameters have been calibrated using market data available on 05/11/2006.

Estimated parameters are:

\[ \theta^* = (18\%)^2, \]
\[ V_0 = (13\%)^2, \]
\[ T = 1.097, \]
\[ k^* = 2. \]

According to (42) we get \( \sqrt{K_{\text{var}_{\text{swp}}}^0} = 16.16\% \), in line with market quotes at the time.

Chart 3 shows how fair strike is affected by model parameters variations. As can be seen from Chart 3, the key driver is the long-run variance. The set of calibrated parameters produces an upward sloping variance swap term structure (fair strikes increase as the time-to-maturity becomes higher), coherently from what is commonly observed on the market.

From our analysis is clear that, for trading purposes, variance swaps are only partially able to capture temporary increases of spot volatility.

Variance swaps will adequately perform only in case of positive shocks of long-term volatility.

![Chart 3: Fair strike sensitivity to model parameters.](image-url)
13 CONCLUSIONS

Prices of listed derivatives don’t reflect the assumptions of simplified pricing models.

Sometimes, errors made with these models are negligible; in other cases there is the need of more accurate models.

In finance, there are not “all purposes” models, suitable for all instruments and valid on every market.

Accurate valuations, well designed hedging strategies and coherent risk analysis can be made only by operators with a deep knowledge of hypotheses and lacks of each model.

Moreover, is very important to assess the main features of financial instruments. A reasonable model for a certain derivative is often not suitable for another one. The comprehension of the key drivers of a specific contract is a crucial step before the use of a specific model.

Factors involved in the choice of a good model (both for analysis and pricing purposes) are multiple. Hereafter will be cited the most relevant: calibration time; stability of calibrated parameters; computational speed; stability and robustness of numerical results; economic meaning of model parameters; availability of risk measures (for example VaR) and sensitivity (greeks).

In the last years, derivative industry made huge progresses. In fact, almost every day, there are new and more complex products on the markets.

In such a variable context, market operators must have a deep knowledge of financial markets coupled with risk analysis skills, fundamental to assess and evaluate risk of complex investments.

REFERENCES


MODELLI PER LA VOLATILITÀ STOCASTICA NELLE SCELTE D’INVESTIMENTO

Riassunto

L’articolo illustra l’utilizzo dei modelli a volatilità stocastica nelle scelte di investimento, che, on daily basis, i trader e i quantitative fund manager sono chiamati a compiere. L’adozione dei modelli a volatilità stocastica sarà giustificata esaminando le principali caratteristiche empiricamente riscontrate nei mercati finanziari in cui detti modelli sono largamente utilizzati. Si examinerà il modello di Heston, in cui la dinamica del sottostante è descritta da un moto geometrico browniano e la varianza da un processo CIR (Cox-Ingersoll-Ross). Dopo averne evidenziato le caratteristiche salienti e i principali limiti (sia teorici che applicativi) si illustreranno le analisi che, tipicamente, sono alla base di scelte d’investimento consapevoli; si farà cenno alle probematiche computazionali ricorrenti in fase di implementazione e saranno, inoltre, forniti spunti per approfondimenti e ulteriori ricerche. Sarà presentato, infine, un caso pratico riguardante lo studio di strumenti derivati in cui la volatilità costituisce un fattore di rilevante importanza.