MODEL ERROR ANALYSIS METHODS

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Abstract

This note reviews some techniques useful for the analysis of specification errors in stochastic models based on diffusion equations or, more in general, on Markov processes, with application in finance

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1. INTRODUCTION

The term 'risk' has been used in the financial community under many different meanings, the stress on each of these changing with events and the evolution of risk management techniques and procedures. With some approximation we can summarize the history of risk management in four phases corresponding to four "names" for risk:

- Market risk
- Credit and counterparty risk
- Operational risk
- Model risk

Market risk received an institutional formalization during the first Basel round. Credit risk required extensive reconsideration in the second Basel round, connected with the dramatic events between 1996 and 1998. More recent and still under discussion is the institutionalization of operational risk management.

On the contrary, model risk is still present like a large unknown in international protocols. It is mentioned as relevant but few suggestions are introduced for dealing with it. On the top of this, relevant theory developments concerning this topic are quite recent and still rather fragmentary. In some sense, model risk still seems to be considered as too difficult a problem, to be dealt on a case by case basis with ad hoc procedures.

This is quite peculiar. In fact model risk, both as model misspecification risk and estimation risk, is implied in all common implementations of risk management. The aim of risk management is to quantify, by the use of measures whose definition is widely discussed in academic literature but quite uniform in practice, some properties of the profit and loss process (e.g. VaR, expected shortfall. etc.). In order to accomplish this, even in the simplest case, probabilistic models are required and these models can only be made operational trough statistical evaluations of their parameters. Since any model is an abstraction, often chosen only for computability reasons, any property of the model not stable to reasonable changes in the model structure cannot be considered as practically relevant. Hence, the ability of and a-priori evaluation of such properties (that is: model risk management) is of paramount relevance for applications.

Actually, a rich toolbox for dealing with model risk already exists, at least in the case where asset prices are described as functionals of diffusions. In this case most modeling is accomplished by specifying partial differential equations (PDEs) of the parabolic type, a kind of mathematical tool extensively studied, both in theory (in mathematics and physics) and as an applied modeling tool (in physics and engineering). Due to the inherent difficulty in analytically solving PDEs, a host of methods for approximating the solutions, both numerically and analytically, have been devised. These methods are not only intended as solution tools, but, most important, as tools useful for studying the stability of models to possible misspecifications. Often these methods, mostly when of analytic nature, are useful not only for computing solutions but also for understanding the implications of models itself.

That such a wealth of methods did develop in time is by no means a puzzle: huge amounts of applied work was done by engineers an physicists using tools from the PDE armoury and in any applied work robustness and approximation are words of trade. Puzzingly, only a small subsection of these tools, mostly the numerical part, did penetrate in fields like mathematical finance and statistics for stochastic processes. The powerful toolbox of analytical methods lies still underused in theory and almost unknown in practice.

Statistics, as a (frequently) very applied field, had to confront similar challenges in time even if with maybe slightly different tools and in different mathematical settings. This could be due to the fact that, in standard statistical settings, the model is specified directly as a class of probability distributions. On the contrary, in stochastic processes, and in particular in Markov processes applications the model is specified by the parameterisation of a partial differential equation or a similar operator. From this it did derive the tendency, in statistics, to consider robustness w.r.t. a contamination of the "true" distribution and not w.r.t. a contamination of the operator describing the process.

In the traditional approach to robustness in statistics, see for instance in (Hampel 1974) and (Huber 1981), the setup is as follows: the statistical model is a known class of distributions but, sometimes, data is observed which does not "come" from the model. In the approach implied by the tools summarised in this paper the statistical model is known only as an approximation, and is specified as a differential operator, while the true model is considered as a perturbed version of some baseline model. The approach is akin to the robust bayesian approach but more specific in that the neighbourhood of the baseline model is explicitly specified and, most important, tools for explicit approximations of specific alternative models in the neighbourhood are provided.

The interest and expediency of the procedures we are going to discuss are so relevant that, significantly, while not directly imported from other fields, cases of these have been rediscovered both in the financial mathematics and statistical literature. The most striking example of this is the recent outpouring of papers on "lack of robustness in economics" originated by (Hansen and Sargent 1995).

The purpose of this paper is to present a short introduction to model risk management trough an analysis of a set of methods available to study model robustness and to provide extensions of computable models to computable approximations of more general, but intractable, models.

2. MODELS IN MATHEMATICAL FINANCE

We only consider stochastic models whose transition probability can be described with a parabolic PDE that is: we deal with diffusions. However, most of what we say in this paper can be extended to general Markov models where an abstract Cauchy problem in terms of the process generator is the standard way for characterizing finite dimensional distributions of the process.

In fact, in the field of Markov processes, contrary to more standard statistical applications, is quite infrequent that the model be specified in terms of probability distribution of observables. The usual specification tool is some difference, differential or integral equation whose parametric form is derived from first principles connected with the problem at hand and whose solution, whose explicit computation is avoided when possible, due to its difficulty, is the required distribution of observables.

The diffusion setting is a very specific contexts which allows a simple characterization of model error as:

1) error in the drift and diffusion function
2) data contamination, that is: contamination of the diffusion process by some other process

For the sake of brevity, we shall consider here only the first kind of error.

Diffusions and PDEs are connected by the concept of martingale. Let the diffusion equation driving the price system be

$$dx = m(x)dt + s(x)dW \tag{1}$$

where for simplicity we suppose *m* and *s* to be time independent processes, and define the generator *A* of the diffusion as

$$A = m(x)\frac{\partial}{\partial x} + \frac{1}{2}s(x)^2\frac{\partial^2}{\partial x^2} \tag{2}$$

Ito's lemma for a suitable *g(x,t)* implies

$$dg(x,t) = (Ag + g_t)dt + g_x s(x)dW \tag{3}$$

A driftless diffusion is a martingale, so $Ag = -g_t$ means g is a martingale. This is an identity if *g* is a known martingale and is a PDE if *g* is unknown. With knowledge on some future value of *g* this becomes

$$Ag = -g_t \tag{4}$$

$$g(x,T) = f(x(T)) \tag{5}$$

This is a Cauchy problem whose abstract version, that is: the same problem with a more general generator *A* characterizes continuous time Markov processes. The solution of this Cauchy problem has three representations

1) martingale (Dynkin formula)

$$g(x,t) = E^A_{x,t}(f(y(T)) \tag{6}$$

2) fundamental solution

$$g(x,t) = \int_Y f(y(T))P(x,t;y,T)dy(T) \tag{7}$$

Where *P* does not depend on *f* and solves

$$AP = -P_t \tag{8}$$

When P is a probability it is the transition function of the process with 'generator' A

3) (formal) operator solution known in Physics as propagator solution

$$g(x,t) = e^{(T-t)A} f \tag{9}$$

where:

$$e^{(T-t)A} f = \sum_{i=0..\infty} \frac{(T-t)^i}{i!} A^i f \tag{10}$$

We only mention the fact that this last representation of the solution can be used for much more general problems than diffusion.

Fundamental for the study of model error is the inhomogeneous PDE

$$Au = -u_t - h(x,t) \tag{11}$$

with the same initial condition as before this is solved (Poisson formula, or Feynman-Kac formula) by

$$u(x,t) = g(x,t) + E^A_{x,t} \int_t^T h(x,v)dv \tag{12}$$

in most practically useful cases we have

$$u(x,t) = g(x,t) + \int_t^T E^A_{x,t}(h(x,v))dv \tag{13}$$

Notice that

$$u(x,t) = E^A_{x,t}(h(x,v)) \tag{14}$$

Solves

$$Au = -u_t \tag{15}$$

$$u(x,v) = h(x,v) \tag{16}$$

3. COMPUTATIONAL NEEDS

In mathematical finance, the valuation and hedging problem for derivative securities and, more in general, the risk management problem is connected with solving, in as much an explicit way as possible, PDEs like these involving the generators A and $A - m(x)\frac{\partial}{\partial x}$. This are the generators corresponding to the so called statistical and risk neutral measure. In fact: no arbitrage prices are martingales (under the risk neutral measure).

However this is not the only role of the above sketched Cauchy problem: transition probabilities, required, for instance, for model simulation, are martingales (in the corresponding measure).

Moreover, parameter estimation typically requires martingales either in the form of likelihoods or in the form of moment conditions. If G is a martingale (statistical measure)

$$E_{t_{i-1}}(g(x_{t_i})) = g(x_{t_{i-1}}) \tag{17}$$

unknown parameters in m and/or s can be estimated minimizing objective functions the like of

$$\sum_{i=2,\dots,n}(g(x_{t_i}) - g(x_{t_{i-1}}))^2 \tag{18}$$

Model choice, model error evaluation and statements about model robustness, require the study of PDEs. Derivative securities pricing and hedging require solutions of PDEs. Calibration and/or estimation of unknown parameters in models require the solution of PDEs.

However, exact solutions of PDEs are uncommon and are mainly restricted to some particular classes of problems (to define this class is a very interesting research topic still only hinted at in the literature on diffusion processes). This has the consequence that modeling tends to be limited to some standard 'solution space'.

Model error risk management is the art of going beyond this.

The toolbox already available to the diffusion modeler for dealing with non analytically solvable models is quite extensive but has, in our opinion, some significant loopholes. There exist powerful techniques for exact numerical solutions. These are often satisfactory for engineering problems, a little less so for physical problems and are, sometimes, under dimensioned for financial problems and more in general, moderate or high dimensional statistical problems. Moreover numerical procedures do not easily yield to model risk analysis: they are more apt for dealing with a single maybe complex model.

Montecarlo methods are widely applied in finance. While quite useful, the power of these methods is often overestimated. In particular, Montecarlo methods are often applied in cases where it is very hard, if even possjùêe, to assess the validity of the conditions for the success of the method. Today is not difficult to point at significant failures of Montecarlo methods both when used in statistics and econometrics, for instance in the form of simulated inference methods and when used in mathematical finance for instance in computing "greeks" (derivatives of option prices to underlying parameters) for some class of even not very exotic options under the standard lognormal hypothesis. This is not to imply that Montecarlo methods have to be avoided as a whole, on the contrary, this is to suggest that such methods are more useful when supplemented by a proper set of alternative techniques useful for controlling and, when necessary, correcting their results.

The widespread use of Montecarlo comes from the practical problem, always present to the mind of the practitioner but often somewhat ignored by the theorist, of obtaining, in reasonable time and with reasonable programming effort, some solution for a given model. From the same need stems forth a second, strictly speaking unacceptable, set of practitioner's procedures. The most common practitioner's operational procedure is that of using computable models, admittedly misspecified, and adapting these in order to reproduce observed data by modifying the values of model parb:%ters. Such (logically inconsistent) procedures bear names the like of: model recalibration, model mark to market, implied parameter evaluation etc. As in the case of Montecarlo, the point is not to deny the usefulness of such procedures which, in many cases, are justified at least by their practical success, but to question their sense and, possibly, to find them some consistent justification.

A more thorough understanding of the power of available modeling tools, requires, at least, a partial answer to the following questions:

- When is it really necessary to solve a model? Practitioners use empirically unsatisfactory models only because they can be solved. Practitioners believe that market calibration is a way out of this problem. Is there sense in this hypothesis? What is the size of the possible error? Does the error decrease if the model is recalibrated? What is the best way to calibrate a model to the market? To fit the price? To fit some greek? Suppose we possess a better model we cannot use because of analytical complexity, is it possible to use this for a better calibration of our operational model? A partial but powerful answer to this problem is given by a set of inequalities which allow the quantification of the consequences of the approximation of a satisfactory but unwieldy model with computable approximations.
- Suppose we evaluate the approximation errors size as too big so that we need a better model. However, for implementation reasons, this should still be a solvable model. Can we define the class of solvable models? This is an interesting and mathematically rather deep problem which has been relevant in PDE theory. It requires and deserves detailed analysis we chose to leave it out of this short expository note.

- Suppose none of these solvable models is found to satisfy our needs, which approximation techniques can be used and which error estimates are available? This paper shall briefly review some classical techniques useful to this aim.

4. MODEL ERROR REPRESENTATION

The formal and obvious first step is that of considering the 'unsolvable' A as decomposed in $B + H$ where B is an operator we can solve. In this case we can represent the solution u^A for $Au^A = -u_t^A$, $u^A(x,T) = f(x(T))$ as

$$u^A = u^B + e \tag{19}$$

where u^B solves

$$Bu^B = -u_t^B \tag{20}$$

$$u^B(x,T) = f(x(T)) \tag{21}$$

and e solves

$$Ae = -e_t - Hu^B \tag{22}$$

$$e(x,T) = 0 \tag{23}$$

Due to the previously summarised results

$$e(x,t) = E^A \int_t^T Hu^B dv \tag{24}$$

This error representation of the model as the expected value of a known function shall be the starting point of what follows. as a first step, in the next section we shall consider a technique for bounding the difference between the exact solution and the approximate solution represented by the operator B.

5. WHY BLACK AND SCHOLES WORKS: INEQUALITIES FOR PDES

The error PDE representation allows for the use of a very interesting set of inequalities of paramount relevance in PDE theory but still only hinted at in finance literature.

The main result can be stated as follows. Under proper regularity conditions (different sets of such conditions exist) if e solves

$$Ae = -e_t - Hu^B \tag{25}$$

$$e(x,T)=0 \tag{26}$$

then, for any suitable PDF Q there exist a $k > 0$ depending on A such that

$$\int_X e^2 dQ \le e^{k(T-t)} \int_t^T \int_X (Hu^B)^2 \, dQdv \tag{27}$$

This is a very simplified version of a more general inequality usually called the “energy” inequality. The relevant point here is that the inequality only depends on the solution of the *B* model so that the bound can be effectively computed in practice.

This inequality gives a possible answer to the reasonable question about the reasons of Black and Scholes (B&S) pricing success even when the B&S model is patently wrong. In fact the inequality implies that a bounded error in the specification of the model parameters imply a bounded error in the model based valuation. Moreover the size of the error can be explicitly evaluated.

Inequalities as this give useful information about the consequence of a specification error and the circumstances under which the model error results is big valuation errors. As an added bonus, in the simple case of european options, they justify the common use of the ’gamma’ of B&S as a measure of model error relevance (the Gamma is the second derivative of the option price w.r.t. the underlying price). In fact, suppose, e.g., that the approximate model is the standard B&S model with volatility s ($dx = sxdW$ in the martingale measure) while the ’true’ model has a generic volatility function S function of the underlying and time, the inequality becomes (for properly defined constants: $\alpha_1, \alpha_2, \beta, \rho$)

$$\int_R e^2 dQ \le \exp\left(\left(\frac{(\alpha_2\rho+\beta)^2}{\alpha_1}+1\right)(T-t)\right) \int_t^T \int_R \left(\frac{1}{2}(s^2-S^2)x^2\Gamma^s\right)^2 dQdv \tag{28}$$

From this formula we see that only state space regions corresponding to sizable values of the gamma (Γ^s) effectively contribute to the error.

Each price greek (the delta or the gamma itself, for instance) solves a PDE and bounds can be derived for it. For this reason the energy inequality also explains in which cases and why hedging with B&S delta and gamma works even if the B&S model hypotheses do not hold.

Among the possible applications of this inequality we mention

- defining and measuring robustness to specification errors
- interpreting the meaning to ’wrong’ greeks as for instance the vega
- suggesting alternatives to ’implied’ procedures for calibrating the parameters of the *b* model with the aim of minimizing the valuation and hedging error

- allowing the calibration of an ’approximate’ model, used for valuation and hedging, to a ’statistical’ model more reasonable for estimation purposes but less amenable to computations.

6. PERTURBATION THEORY

Perturbation theory is a powerful tool in modeling with PDEs which. While central in the fields of Physics and Engineering, perturbation theory did receive little or no interest in the stochastic processes literature. This is quite surprising and it could be very interesting to understand the reasons underlying this event. The following extract from (Hille and Phillips 1957) (Ch. XIII, page. 388) is useful in order to clarify the relevance of the method and to stress the reason why a perturbation analysis should be operated on any model explicitly introduced as an approximation. The interested statistician should compare this with the introductory section in standard texts on robust statistics as, for instance, (Hampel and Stahel 1986).

“… Perturbation theory has long been a useful tool in the hands of the analyst. It is used to determine the state of a system which is in a certain sense close to a known system. In our case the known system is a semi-group $T(\xi; A_0)$ of linear bounded operators with infinitesimal generator A_0 and we wish to ascertain that nearby operators likewise generate semi-groups. Moreover it is desirable that the semi-group $T(\xi; A)$ vary continuously with A, for in this case the problem of generation discussed in Chapter XII will be “well set.”

We shall call a semi-group property stable if it holds for all semi-groups $T(\xi; A)$ with A sufficiently close to A_0 whenever it holds for $T(\xi; A_0)$. Of course, in order to make this notion precise we shall have to define a topology for the set of infinitesimal generators.

In applications one would expect that the infinitesimal generator itself is known only to within certain limits of error and hence that a physical meaning could be attached only to the stable properties of the associated semi-groups. Mathematically one would expect that the stable properties of a semi-group are more basic than the others and that the significant theorems in the subject would evolve about these properties...”

There exist many perturbative methods. I briefly summarize here the so called regular perturbation method. With this method a given, unsolvable, PDE is first written, if possible, of some perturbed version of a solvable PDE. The solution is then expressed as a power series (in the perturbation parameter) of functions derived as explicit solutions of modifications of the solvable PDE.

Suppose we wish to solve

$$\frac{1}{2}\left(s(x)^2+\varepsilon k(x)^2\right)\frac{\partial^2}{\partial x^2}u+u_t=0 \tag{29}$$

$$u(x,T)=f(x) \tag{30}$$

where ε is a control parameter and $k(\cdot)$ is a perturbation term.

Suppose that the solution of this problem is available if $\varepsilon = 0$. and that the solution of the perturbed problem actually exists for some neighbourhood of $\varepsilon = 0$. Notice that, if this is false and if the perturbation $\varepsilon k(\cdot)$ represents a reasonable and realistic variant of the basic model, any further use of the solvable model could be questioned as unstable when "reasonably" perturbed (see the quotation above).

If this solution exists, the second step is to state whether some continuity in ε holds. Solutions could exist for an interval of values of ε but these solutions could wildly vary w.r.t. changes of ε. Again, in this case the actual use of the model is under question.

Suppose both problems are solved in the positive. In the usual setting we are unable to compute the solution for $\varepsilon \neq 0$ but we can solve the PDE for $\varepsilon = 0$. Is there any way of extending this solution in order to approximate the exact solution for $\varepsilon \neq 0$?

A possible answer lies in a proof that the solution can be written as a power series in ε. Obviously, to give a proof of this fact is a way for answering in the positive to the above continuity questions and, most relevant, it opens the possibility for simple approximation algorithms.

In fact, if

$$u(x,t)=\sum_{i=0}^{\infty}\varepsilon^i u_i(x,t) \tag{31}$$

then each term of this series solves a PDE which depends on previous terms and known functions. All terms can be computed if the zero order term can be computed and the zero order term solves the equation with $\varepsilon \neq 0$.

$$\frac{1}{2}s(x)^2\frac{\partial^2}{\partial x^2}u_0+u_{0t}=0 \tag{32}$$

$$u_0(x,T)=f(x) \tag{33}$$

If *s(x)* = *sx* as in the B&S case the first term is simply the B&S formula and can be represented as $u_0 = E_t^B(f(x))$, where the expected value is computed using the fundamental solution of $1/2s^2x^2\partial^2/\partial x^2 u_0 = -u_{0t}$ that is: the lognormal distribution.

The term of order 1 solves

$$\frac{1}{2}s(x)^2\frac{\partial^2}{\partial x^2}u_1 + u_{1t} = -\frac{1}{2}k(x)^2\frac{\partial^2}{\partial x^2}u_0 \tag{34}$$

$$u_1(x,T) = 0 \tag{35}$$

This is a non homogeneous equation whose solution, as we have seen, is, setting $Hu_0 = \frac{1}{2}k(x)^2\frac{\partial^2}{\partial x^2}u_0$

$$u_1 = \int_t^T E_t^B(Hu_0(v))dv \tag{36}$$

When B is the B&S operator, the term $E_t^B(Hu_0(s))$ is the value at t, computed in the lognormal B&S world, of an option expiring at time v with payoff $Hu_0(v)$.

For instance: if $H = \frac{1}{2}k(x)^2\frac{\partial^2}{\partial x^2} = 1/2(S^2 - s^2)x^2\partial^2/\partial x^2$, as in the energy inequality example, we have

$$E_t^B(Hu_0(v)) = 1/2E_t^B((S^2 - s^2)x^2\Gamma_{B\&S}(v)) \tag{37}$$

In general the i-th term solves

$$Bu_i + u_{it} = -Hu_{i-1} \tag{38}$$

that is

$$\frac{1}{2}s(x)^2\frac{\partial^2}{\partial x^2}u_i + u_{it} = -\frac{1}{2}k(x)^2\frac{\partial^2}{\partial x^2}u_{i-1} \tag{39}$$

$$u_i(x,T) = 0 \tag{40}$$

And the solution is

$$u_i = \int_t^T E_t^B(Hu_{i-1}(v))dv \tag{41}$$

i =1, 2,…

As we stated above, this is true if the solution of the perturbed model can be developed in power series for ε. The proof of this point can be delicate, a general approach can be found in (Kato 1980). In a forthcoming paper (Carta and Corielli 2006) a more specific proof tuned to financial valuation problems is developed.

In the self adjoint case the perturbation method is directly connected to the sieve method, a numerical technique for the estimation of parameters in reversible Markov processes (Darolles and Gourieroux 2001), (Hansen and Touzi 1998).

7. SMALL TIMES

The classic operator interpretation of the autonomous PDE

$$Au = -u_t \tag{42}$$

with $u(x,T) = g(x(T))$ is

$$u(x,t) = e^{(T-t)A} g \tag{43}$$

where

$$e^{(T-t)A} = \sum_{i=0}^{\infty} \frac{(T-t)^i}{i!} A^i \tag{44}$$

This series is typically difficult to use and is only formally defined as the smoothness requirements on g necessary for its well definition are usually not valid. There are many ways off the problem. A very simple trick is similar to the starting point of a perturbative expansion.

Suppose $A = B + H$ and we want to solve

$$(B+H)u = -u_t \tag{45}$$

$$u(x,T) = g(x) \tag{46}$$

but we can solve this only if $H = 0$ and we call this solution u^B. In this case, by simple algebra, we find

$$u = u^B + e \tag{47}$$

$$(B+H)e = -e_t - Hu^B \tag{48}$$

$$e(x,T) = 0 \tag{49}$$

So that, by Dynkin formula

$$e(x,t) = E^A_{x,t} \int_t^T Hu^B(X,v)dv = \int_t^T E^A_{x,t} Hu^B(X,v)dv \tag{50}$$

But $E^A_{x,t} Hu^B(X,v) = h(x,t;v)$ solves

$$(B + H)h = -h_t \tag{51}$$

$$h(X,v;v) = Hu^B(X,v) \tag{52}$$

typically this h is better behaved than $g(x)$ and the expansion

$$u = u^B + \int_t^T \sum_{i=0}^{\infty} \frac{(v-t)^i}{i!} (B+H)^i h(X,v;v)dv \tag{53}$$

is well defined.

The standard application of such a result is this: g is, say, the call payoff. B is the B&S operator, H a perturbation of this and, typically, $Hu^B(X,v)$ is some function of x times the gamma of the B&S option.

8. PARAMETRIX SERIES

The parametrix series represents a very general way for expressing the fundamental solution of a parabolic PDE as a series of known functions. It can be applied both to the forward and backward equation. As is true in the case of the perturbation expansion, the effective computability of the parametrix series only requires the computability of some 'zero order' term.

A very interesting point of the parametrix series is that, when properly truncated, it gives representations of the solution which both justify and extend standard widely observed practitioner behaviours as volatility smile fitting and hedging with tweaked greeks. Moreover, as in the case of perturbation theory, the errors due to truncation are expressed as option payoffs so that, at least in principle, the cost of model error becomes equivalent to the cost of specific options. As a last relevant point we stress the fact that the parametrix method allows for an easy truncation error valuation.

As is well known, under some hypotheses, a parabolic PDE can be solved in two ways: directly (forward parametrix) or in its adjoint form (backward parametrix) (see e.g. (Friedman 1964)). Both solutions are interesting and shall be briefly described in what follows

Backward Parametrix Consider the differential operator

$$L = \sigma(x,t)\partial_{xx} - \partial_t, \tag{54}$$

with fundamental solution $\Gamma(z,\zeta)$ ($z \equiv (x,t) \simeq$ "start" and $\zeta \equiv (X,T) \simeq$ "end"). Suppose

$$\Gamma(z;\zeta) \quad = \quad P(z;\zeta) + \text{"correction term"} \tag{55}$$

where $P(z;\zeta)=\Gamma_z(z;\zeta)$ and Γ_w is the fundamental solution to

$$L_w=\sigma(w)\partial_{xx}-\partial_t \tag{56}$$

NB: L_w is a constant coefficient operator where $\sigma(x,i)$ is 'frozen' at *w*. The fundamental solution of the frozen operator (parametrix) is a Gaussian with volatility depending on the starting point. Hence Γ_w is known explicitly (Black&Scholes formula).

Let us now look for Γ in the form

$$\Gamma(z;\zeta)=P(z;\zeta)+\int P(z;w)\Phi(w;\zeta)dw \tag{57}$$

being $L\Gamma=0$, the unknown function Φ satisfies

$$0=LP(z;\zeta)+\int LP(z;w)\Phi(w;\zeta)dw-\Phi(z;\zeta) \tag{58}$$

We then find the recursive formula

$$\Phi(z;\zeta)=LP(z;\zeta)+\int LP(z;w)LP(w;\zeta)dw+\ldots \tag{59}$$

Forward Parametrix

Consider the differential operator

$$\tilde{L}=\sigma(x,t)\partial_{xx}+\partial_x\sigma(x,t)\partial_x+\partial_{xx}\sigma(x,t)+\partial_t, \tag{60}$$

with fundamental solution $\tilde{\Gamma}(\zeta;z)=\Gamma(z;\zeta)$ ($z\equiv(x,t)\simeq$ "start" and $\zeta\equiv(X,T)\simeq$ "end")

Now set

$$\tilde{\Gamma}(\zeta;z) \quad = \quad P(z;\zeta)+\text{"correction term"} \tag{61}$$

where $P(z;\zeta)=\Gamma_\zeta(z;\zeta)$. That is: the parametrix is a gaussian but with a different volatility for any terminal *X*.

Let now *C* be the price of the option with payoff *h*

$$C(t,S)=\int\Gamma(t,S;T,x)h(x)dx=\sum_{n=1}^{\infty}C^{(n)}(t,S) \tag{62}$$

The first term is going to be the Black&Scholes price with $\sigma=\sigma(t,S)$

$$C^{(1)}(t,S)=\int P(t,S;T,x)h(x)dx \tag{63}$$

The n-th term shall be the Black&Scholes price with $\sigma=\sigma(t,S)$ and transaction cost $LC^{(n-1)}$

$$C^{(n)}(t,S)=\int P(t,S;\zeta)LC^{(n-1)}(\zeta)d\zeta \tag{64}$$

For the forward parametrix the expansion is similar to the backward case but the first term considers a 'Black and Scholes' formula with a X dependent volatility. this implies an interesting new interpretation of volatility smile fitting: the standard way uses a different gaussian for each strike, the backward parametrix way uses the same (nongaussian) distribution (it can be normalized) but with volatility dependent on X.

Both cases are interesting: the backward one is more intuitive as it starts from the B&S formula. Moreover, while the first term is the standard B&S formula, the point where the volatility is frozen depends on the current price of the underlying. We then have that the delta is given by

$$\Delta_{BS}+Vega_{BS}\sigma_x \tag{65}$$

This correction is by far not unknown to practitioners who compute the smile corrected delta.

The forward case is still more intriguing as it gives a formal interpretation of a standard practitioner habit i.e. the skew correction in pricing. In standard practice a different volatility (i.e.) a different risk neutral distribution) is used for valuing options with different strikes. The first term of the forward parametrix expansion suggest an alternative, and perhaps more coherent, procedure: use a different volatility for each terminal value of the underlying. In this case the approximation of therisk neutral density is the same for all strikes but, instead of being a gaussian is a modified gaussian (it can be normalized) where the volatility is a function of the terminal value of the underlying.

We end this section by noticing that the parametrix method allows for easy to compute explicit truncation error bounds of obvious practical usefulness.

9. SOME CONCLUSIONS

We summarize the content of this paper in the following points.

- Model error risk management requires measures of the consequences of model error. From the implementation point of view this requires analytical approximations.

- In the case of diffusion (or, more general) Markov processes, the model risk problem admits a rigorous approach based on classical tools for sensitivity analysis of solvable models. These tools are somewhat overlooked in standard textbooks and courses of stochastic processes and financial mathematics which are typically based on a stochastic calculus approach and less on a PDE approach.
- Solvable models can be used as a basis for the study of perturbed classes of models. Approximate solutions can be expressed as series expansions requiring only the solution of the ’zero order’ model.
- There is a need for identifying stable and non stable properties. This is required not only for numerical solution purposes but also for assessing the reliability of estimation methods.
- There are obvious connections with statistical robustness in the sense of robustness w.r.t. a class of models as opposed to robustness w.r.t. data contamination.
- Many ad hoc and apparently inconsistent manipulations of basic computable models done by practitioners can be seen as intuitive use of rigorous approximations (both of the perturbative and the parametrix kind).

10. BIBLIOGRAPHICAL NOTES AND THANKS

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The topics summarized in this article are considered in depth in three papers: (Corielli 2006), (Carta and Corielli 2006) and (Corielli and Pascucci 2006).

The standard treatment of PDEs and diffusions, under classical hypotheses, can be found in chapter 6 of (Friedman 1976). A standard reference on semigroups and Markov processes is: (Dynkin 1965). The standard text on perturbations is Tosio Kato wonderful book: (Kato 1980) which completes the general analysis of perturbed linear operators already discussed in the classic (Hille and Phillips 1957). A detailed analysis of the classical form of the parametrix method can be found in (Friedman 1964).

Claudio Albanese and Coauthors, (http://www.level3finance.com) develop two related lines of work: first the extension of the class of computable models by a suitable class of transforms of classical models, second the extensive use of space-only discretization in autonomous models.

The sieve method (in practice space-only approximation of generators) is applied to the estimation of diffusions in (Darolles and Gourieroux 2001). A summary of classical results on eigendecompositions of (mainly self adjoint) generators to parameter estimation can be found in (Hansen and Touzi 1998). A recent and quite promising paper on small time approximation is (Mele and Kristensen 2006).

Two interesting papers representing an early use of perturbation theory in finance are: (Samuelson 1970) and (Samuelson and Merton 1974). These two papers can be considered as the introduction (in modern times) to the problem of robustness in economics. The perturbed object is the utility function. The first paper is based on the consequences of general convexity properties of the utility while the second performs a real perturbative analysis with some approximation results. Both papers follow the approach suggested here that is, they do not endogenize model (utility) uncertainty.

REFERENCES

CARTA P. and CORIELLI F. (2006) On analytic perturbations of the b&s pde, *W.P. Submitted*.

CORIELLI F. (2006) Hedging with energy, *Mathematical Finance*, 495–517.

CORIELLI F. and PASCUCCI A. (2006) Parametrix approximations for option prices, *W.P. Submitted*.

DAROLLES S. and GOURIEROUX C. (2001) Truncated dynamics and estimation of dffusion equations, *Journal of Econometrics*, 102, 1–22.

DYNKIN E. (1965) *Markov Processes, Vols. 1-2*, Springer-Verlag, Berlin.

FRIEDMAN A. (1964) *Partial Differential Equations of Parabolic Type*, Prentice-Hall, New York.

FRIEDMAN A. (1976) *Stochastic Differential Equations and Applications, Vols. 1-2*, Academic Press, New York.

HAMPEL F. RONCHETTI E.R.P. and STAHEL W. (1986) *Robust Statistics. The Approach Based on Influence Functions*, John Wiley and Sons, New York.

HAMPEL F.R. (1974) The influence curve and its role in robust estimation, *Journal of the American Statistical Association*, 69, 383–393.

HANSEN L. P. SCHEINKMAN J.A. and TOUZI N. (1998) Spectral methods for identifying scalar diffusions, *Journal of Econometrics*, 86, 1–32.

HANSEN L. and SARGENT T. (1995) Discounted linear exponential quadratic gaussian control, *IEEE Transactions on Automatic Control*, 40, 968–971.

HILLE E. and PHILLIPS S. (1957) *Functional analysis and semigroups*, A.M.S Colloquium publications.

HUBER P.J. (1981) *Robust Statistics.*, John Wiley and Sons, New York.

KT. (1980) *Perturbation theory for linear operators*, Springer-Verlag, Berlin.

MELE A. and KRISTENSEN D. (2006) Adding and subtracting black-scholes: A new approach to approximating derivative prices in continuous time asset prices models, *London School of Economics W.P.*

SAMUELSON P.A. (1970) The fundamental approximation theorem of portfolio analysis in terms of means variances and higher moments, *Rew. Econ. Studies*, 37.

SAMUELSON P.A. and MERTON B. (1974) Generalized mean variance tradeoffs for best perturbation corrections to approximate portfolio decisions, *J. of Finance*, 29.

METODI PER L'ANALISI DI ERRORI DI MODELLO

Riassunto

Questa nota passa in rassegna alcune tecniche per l'analisi di errori di specificazione in modelli stocastici basati su equazioni di diffusione, o più in generale su processi markoviani, con applicazione in finanza.