STOCHASTIC MORTALITY IN LIFE INSURANCE[1]

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Abstract

The assessment of the risk profile of a life insurer (or a pension fund) requires an explicit valuation of the random number of deaths in a given population, and thus a stochastic approach to the representation of mortality. This paper is meant as a review of the basic models suitable to represent the risk of "random fluctuations" (or "process risk") and the risk of "systematic deviations" ("uncertainty risk", with particular attention to longevity risk), as well as to assess the impact of such risks on the technical performance of the life portfolio.

Keywords: mortality risks, random fluctuations, systematic deviations, longevity risk.

1. INTRODUCTION

The awareness of advantages in respect of insurance risk management provided by "large" portfolio sizes can be traced back to the end of the 18th century. In 1786 Johannes Tetens addressed the analysis of mortality risk inherent in an insurance portfolio. In his analysis, the evidence of the role of $\sqrt{n}$ in determining the riskiness of a portfolio, where n denotes the number of policies in the portfolio itself, emerges. In particular, as pointed out by Haberman (1996), Tetens showed that the risk in absolute terms increases as the portfolio size n increases, whereas the risk in respect of each insured decreases in proportion to $\sqrt{n}$. In a modern perspective, Tetens' ideas constitute a pioneering contribution to the individual risk theory.

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Teten's findings about the "small" variability in the insurer's payout when the portfolio size is reasonably "large" justify the deterministic approach to life insurance calculations. Even if the so called biometric functions, e.g. q_x, l_x, etc., involved in actuarial models, are correctly meant as probabilities, expected values, etc., when these quantities are only used to calculate expected values we actually refer to a deterministic approach, as no measure of riskiness is allowed for.

Conversely, a stochastic approach to actuarial calculations requires focussing on random variables and related probability distributions. In particular, an appropriate tool is provided by the random residual lifetime of an individual aged x, T_x, and the relevant probability distribution, often assigned in terms of the survival function (referred to the random total lifetime T_0), $S(t) = \Pr\{T_0 > t\}$. The expression of the random present value (e.g. at policy issue) of insured benefits as a function of the residual lifetime T_x (x being the age at policy issue) is due to de Finetti (1950, 1957) and Sverdrup (1952), and constitutes the starting point of a stochastic approach based on individual lifetimes.

A stochastic approach as describedabove allows for the so called"process risk", i.e. the risk arising from randomness of the individual lifetimes around the relevant expectations, and hence from random fluctuations of the insurer's payout. Actually, the probabilistic structure (e.g. the survival function $S(t)$) is assumed to provide a "good" description of randomness in policyholders' lifetimes and hence in portfolio results. However, some degree of uncertainty may affect the choice of the survival function. If a "law" (e.g. Gompertz, Makeham, Thiele, etc.) has been chosen to describe the age pattern of mortality, uncertainty may concern the type of function or its parameter values. In the former case, the "model risk" arises, while in the latter the "parameter risk" is involved. In both cases, the risk is usually referred to as the "uncertainty risk".

While the process risk should be carefully accounted for in particular when small insurance portfolios (or pension funds) are dealt with, the uncertainty risk has a dramatic importance especially in relation to large portfolios (or pension funds). Actually this risk cannot be lowered increasing the portfolio size, since it concerns the portfolio as an aggregate risk. An important example of uncertainty risk is given by the "longevity risk" arising from uncertainty in future mortality trends, and possibly leading to underestimation of insurer's liabilities when benefits consist in life annuities.

Randomness of the lifetimes and/or the number of deaths in a given population must be explicitly allowed for while assessing the risk profile of an insurer (or a portfolio fund). Hence a stochastic approach to mortality is needed. This paper deals with stochastic models which can be used to represent the risk of random fluctuations (i.e. the process risk) and the risk of systematic deviations (the uncertainty risk and, in particular, the longevity risk) in mortality, and to assess the relevant impact on technical results.

The paper is organized as follows. Section 2 deals with the representation of mortality in actuarial models for life insurance and annuities. Section 3 focuses on the process risk in a portfolio of policies providing death benefits; a "static" mortality assumption is adopted, in the sense that no mortality trend is allowed for. Conversely, Section 4 deals with the uncertainty risk in a "dynamic" framework, looking in particular at the impact of longevity risk in a portfolio of life annuities. Section 5 concludes the paper with some final remarks.

2. ALLOWING FOR MORTALITY IN LIFE INSURANCE MODELS

Future cash flows (premium income, benefit outgo, expenses, etc.), portfolio reserves, profits, etc., constitute examples of random quantities related to a life insurance portfolio. When assessing the risk profile of an insurance portfolio (or a pension fund) the first step consists in singling out "causes" and "components" of risks.

2.1 RISKS IN LIFE INSURANCE RESULTS

Causes of risk in a life insurance portfolio relate to financial aspects (e.g. yield from investments, inflation, etc.), demographical aspects (e.g. random lifetimes of policyholders, disability, lapses and surrenders, etc.), expenses. We now deal with demographical aspects only, focussing in particular on policyholders' random lifetimes, which in turn determine the random death frequency in a portfolio.

Random frequencies are usually summarized in terms of expected frequencies, estimated from statistical data and possibly allowing for future trends (especially when long-term contracts are concerned). From expected frequencies of death other quantities can be derived, such as expected cash flows, expected portfolio reserves, expected profits, etc.

Clearly, expected values can effectively summarize quantitative aspects of a portfolio provided that risks have a limited impact on the portfolio results. Referring to mortality risks, this is the case, for example, of a very large portfolio with a significant statistical experience about mortality, whence no uncertainty concerning the demographical basis affects pricing and reserving.

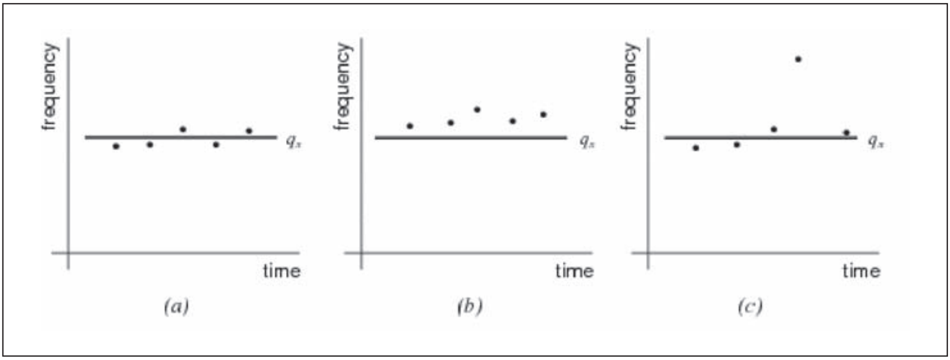

Fig. 1: Mortality in a static context: expectation vs experience.

Figures 1*(a)*, 1*(b)* and 1*(c)* show the expected mortality rate q_x at some given age x (the solid line), and three sets of possible future mortality experience (the dots). Deviations from the expected mortality rate in Figure 1*(a)* can be reasonably explained in terms of random fluctuations of the outcomes (the observed mortality rates) around the relevant expected value q_x .

The risk of random fluctuations is a well-known "component" of risk in the insurance business, in both life and non-life insurance areas. As already mentioned, it is often named "process risk". Fundamental results in risk theory state that the severity of this risk (conveniently measured) decreases as the portfolio size increases (see Section 1). For this reason, the random fluctuation risk is a "pooling risk".

The experienced profile depicted in Figure 1*(b)* can be hardly attributed to random fluctuations only. Much more likely, this profile can be explained as the result of an actual mortality different from the expected one. So, systematic deviations arise. The risk of systematic deviations can be thought of as a "model risk" or a "parameter risk" (as mentioned in Section 1), referring to the model used for representing mortality and the relevant parameters. The expression "uncertainty risk" is often used to refer to model and parameter risk jointly, meaning uncertainty in the representation of a phenomenon (e.g. the mortality).

The risk of systematic deviations cannot be hedged by increasing the portfolio size, since deviations concern all the insureds or annuitants in the same direction. For this reason, the systematic deviation risk is called a "non-pooling risk".

The experienced mortality profile depicted in Figure 1*(c)* likely represents the effect of the "catastrophic" component of mortality risk, namely the risk of a sudden and short-term rise in the mortality frequency, for example because of an epidemic or a natural disaster.

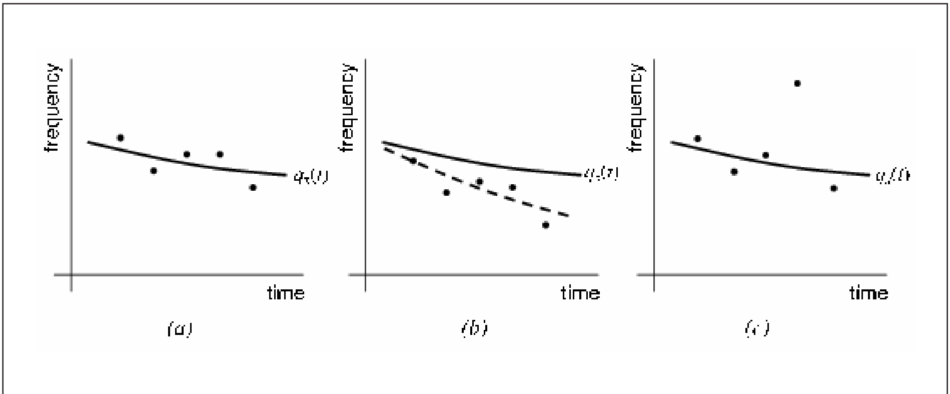

Fig. 2: Mortality in a dynamic context: expectation vs experience.

Figures 2*(a)*, 2*(b)* and 2*(c)* show similar situations, referring however to a dynamic mortality context, in which it is assumed that a decline in mortality will occur in the future and hence expected mortality is calculated via an appropriate projection model.

2.2 RISK DIVERSIFICATION

As mentioned in Section 2.1, the severity of random fluctuations risk decreases as the portfolio size increases. Hence, this component of the mortality risk can be naturally "diversified" inside the insurance process. In particular, the risk can be (partially) ceded to a reinsurer, whose larger portfolio can better face the risk itself.

Diversification of the risk of systematic deviations in mortality only relies (inside the insurance process) on appropriate hedging techniques (see Olivieri and Pitacco (2006) and the references therein). Examples of hedging are as follows:

- insurance products combining benefits in case of life (e.g. a life annuity) with benefits in case of death (e.g. the payment of part of the mathematical reserve to the beneficiaries);
- portfolios consisting of both policies with life benefits only and policies with death benefits only, with the proviso that the insureds' age ranges are similar and/or that the same mortality trend affects all the age ranges involved.

Further, the systematic deviation risk can be diversified transferring (part of) the risk itself to the financial markets, typically via longevity bonds.

Finally, the catastrophe risk can be diversified avoiding dramatic concentrations of insured risks in small geographical areas.

2.3 DETERMINISTIC MODELS AND STOCHASTIC MODELS

When the probabilistic description of the age pattern of mortality (in terms of given probabilities of death and survival, force of mortality, etc.) is only used to derive expected values, a “deterministic” model is adopted. In this context, riskiness can be (roughly) allowed for via the use of “prudential” demographical bases, i.e. including a safety loading in premiums and reserves. It should be stressed that, frequently, the magnitude of the safety loading does not rely on a sound risk assessment.

An appropriate alternative is provided by stochastic models. A stochastic model consists of the following items:

- probabilistic assumptions concerning individual lifetimes (in terms of probabilities of death and survival, or survival functions, forces of mortality, etc.);
- correlation assumptions among random lifetimes (albeit the independence hypothesis is commonly accepted);
- possibly, alternative mortality scenarios in order to deal with systematic deviations.

A stochastic model provides outputs such as the probability distributions of portfolio results (cash flows, profits, etc.) and in particular single-figure indexes (expectations, variances, percentiles, etc.).

In some cases (for example, see Section 3) the distribution of the annual random number of deaths in a portfolio can be directly calculated, possibly via some approximation (instead of dealing with the individual random lifetimes).

Stochastic models can be used in both a static framework (i.e. not allowing for future mortality trends) and in a dynamic framework, where alternative mortality scenarios can represent uncertainty in future mortality trends. The following Sections illustrate the use of stochastic models in a static context allowing for random mortality fluctuations only (Section 3) and, respectively, in a dynamic one in which both random fluctuations and systematic deviations, viz. the longevity risk, are dealt with (Section 4).

3. “STATIC” MORTALITY

In this Section we refer to a portfolio of one-year insurance covers only providing a death benefit. In practice, such a portfolio can represent a group insurance, or a one-year section of a more general portfolio consisting of policies with a positive sum at risk due to the presence of some death benefit.

Many applications can be envisaged in the field of insurance risk management. For example:

- assessment of the safety loading to be adopted when pricing and/or reserving;
- choice of reinsurance arrangements;
- capital allocation for solvency purposes.

3.1 THE PORTFOLIO PAYOUT FOR ONE-YEAR DEATH COVERS

Let n denote the number of insureds, and C_j the sum assured for the j-th contract ($j = 1, 2, \ldots, n$). The individual random payout is given by

$$Y_j = \begin{cases} C_j & \text{if the } j\text{-th insured dies in the year} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Hence, the portfolio random payout is as follows:

$$Y = \sum_{j=1}^{n} Y_j \tag{2}$$

Assume that the individual lifetimes are independent, whence the independence of the random variables Y_j follows. Further, if we assume $C_j = C$ for $j = 1, 2, \ldots, n$, and the same probability of dying q for all the insureds, Y has $0, C, \ldots, nC$ as the possible outcomes, with the binomial probability distribution:

$$\Pr\{Y = hC\} = \binom{n}{h} q^h \, (1-q)^{n-h} \tag{3}$$

In more general situations the (exact) distribution can be calculated via recursion formulae (see for example Panjer and Willmot (1992)).

In actuarial practice, various approximations to the exact distribution of the random payout are frequently used. In particular (see Panjer and Willmot (1992)):

- if $C_j = C$ for $j = 1, 2, \ldots, n$, the use of the Poisson distribution relies on the Poisson assumption for the annual number of deaths;
- for more general portfolios, the compound Poisson model is adopted;
- in general, the normal approximation is frequently used.

Whatever the approximating distribution may be, the goodness of the approximation must be carefully assessed, especially with regard to the right tail of the distribution itself, as this tail quantifies the probability of large losses.

3.2 EXAMPLE 1

Data are as follows:

- sum assured $C_j = 1$ for $j = 1, 2, \ldots, n$;
- probability of death $q = 0.005$;
- portfolio sizes: $n = 100$, $n = 500$, $n = 5000$.

The (exact) binomial distribution and the normal approximation have been adopted for $n = 500$ and $n = 5000$; the (exact) binomial distribution and the Poisson approximation have been used for $n = 100$. Figures 3 and 4 and Tables 1 to 3 show numerical results.

The following aspects should be stressed. In relation to portfolio sizes $n = 500$ and $n = 5000$, the normal approximation tends to underestimate the right tail of the payout distribution (see Table 1). Conversely, the Poisson distribution provides a good approximation to the exact distribution (also for $n = 100$; see Tables 2 and 3); unlike the normal approximation, the Poisson model tends to overestimate the right tail, whence a conservative assessment of liabilities follows.

Tab. 1: Right tails of the binomial (exact) distribution and the Normal approximation.

y	$n = 500$ $\Pr\{Y > y\}$		y	$n = 5000$ $\Pr\{Y > y\}$	
	Binomial	Normal		Binomial	Normal
5	0.041602820	0.056471062	30	0.136121887	0.158048811
6	0.013944069	0.013238288	35	0.022173757	0.022480517
7	0.004135437	0.002164124	40	0.001983179	0.001316908
8	0.001097966	0.000244022	45	0.000101743	3.03545E-05
9	0.000263551	1.88389E-05	50	3.13201E-06	2.68571E-07
10	5.76731E-05	9.90663E-07	55	6.02879E-08	8.9912E-10
…	…	…	…	…	…

Tab. 2: Binomial (exact) distribution and Poisson approximation (n = 100).

y	$Pr\{Y = y\}$	
	Binomial	Poisson
0	0.605770436	0.606530660
1	0.304407255	0.303265330
2	0.075719392	0.075816332
3	0.012429649	0.012636055
4	0.001514668	0.001579507
5	0.000146139	0.000157951
6	1.16275E-05	1.31626E-05
7	7.84624E-07	9.40183E-07
8	4.58355E-08	5.87614E-08
9	2.35447E-09	3.26452E-09
10	1.07667E-10	1.63226E-10
…	…	…

Tab. 3: Right tails of the binomial (exact) distribution and the Poisson approximation (n = 100).

y	$Pr\{Y > y\}$ Binomial	Poisson
3	0.001673268	0.001752
4	0.000158599	0.000172
5	1.24604E-05	1.42E-05
6	8.32926E-07	1E-06
7	4.83022E-08	6.22E-08
…	….	…

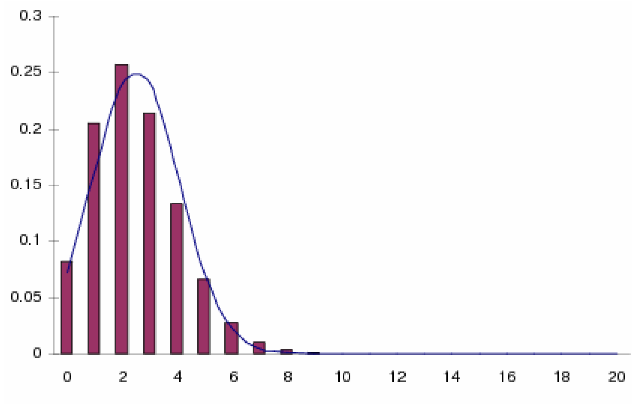

Fig. 3: Probability distribution of the random payout (n= 500). Binomial (exact) distribution and Normal approximation.

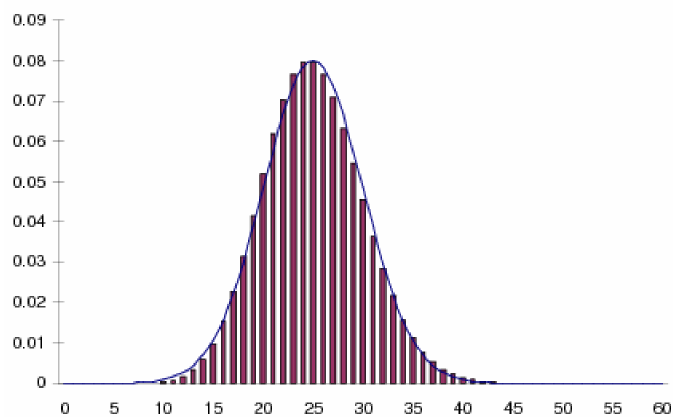

Fig. 4: Probability distribution of the random payout (n= 5000). Binomial (exact) distribution and Normal approximation.

3.3 EXAMPLE 2

Consider the portfolio structures described in Table 4. Probability of death is still q = 0.005 for all insureds. Assume alternatively n =100, n = 500, n = 5000 as the portfolio size.

Tab. 4: Portfolio structures.

Number of policies	Sum assured
$0.5n$	1
$0.3n$	3
$0.2n$	5

The exact distribution (calculated via recursion formulae) and the compound Poisson approximation have been adopted for n =100; the normal approximation and the Poisson approximation have been used for n =5000 and . Figures 5 and 6 and Tables 5 and 6 show numerical results.

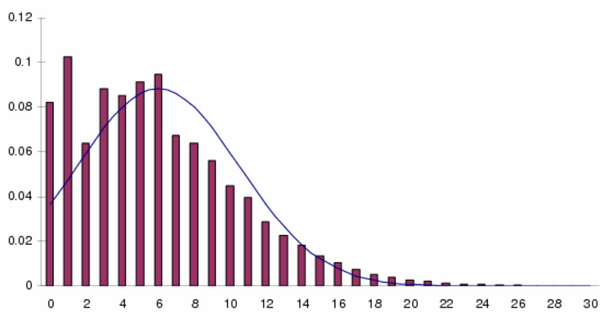

Fig. 5: Probability distribution of the random payout (n = 500). Compound Poisson approximation and Normal approximation.

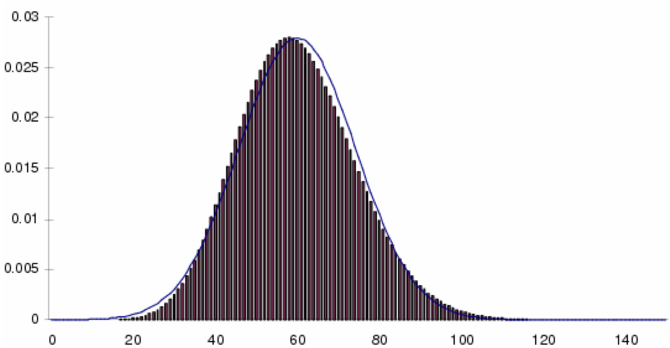

Fig. 6: Probability distribution of the random payout (n = 5000). Compound Poisson approximation and Normal approximation.

Tab. 5: Exact distribution and compound Poisson approximation (n = 100).

y	$\Pr\{Y = y\}$ Exact distribution	Compound Poisson
0	0.60662	0.606530660
1	0.15324	0.151632665
2	0.01917	0.018954083
3	0.09140	0.092559106
4	0.02384	0.022843619
5	0.06276	0.063501114
6	0.02134	0.022223868
7	0.00377	0.003616091
8	0.00932	0.009469885
9	0.00259	0.002643336
10	0.00335	0.003403863
11	0.00143	0.001474941
12	0.00029	0.000280523
13	0.00052	0.000487447
14	0.00013	0.000150518
…	…	…

Tab. 6: Right tails of the exact distribution and the compound Poisson approximation (n = 100).

y	$Pr\{Y > y\}$ Exact distribution	Compound Poisson
5	0.04297	0.043979
6	0.02163	0.021755
7	0.01786	0.018139
8	0.00854	0.008669
9	0.00595	0.006026
10	0.0026	0.002622
…	…	…
15	6E-05	0.000104
16	5E-05	4.21E-05
17	3E-05	2.9E-05
18	2E-05	1.22E-05
…	…	…

The following aspects should be stressed. For $n = 100$, the compound Poisson distribution offers a good approximation to the exact distribution (see Table 5), with a slight overestimation of the right tail, but for “extreme” outcomes. For not very large portfolios (e.g. the case $n = 500$) the normal approximation does not capture the peculiarity of the payout distribution due to the portfolio structure (see in particular Figure 5). For large portfolio sizes (e.g. $n = 5000$) the normal distribution is "biased" if compared to the compound Poisson, leading in particular to an underestimation of the right tail (see Figure 6).

4. "DYNAMIC" MORTALITY

In many countries, mortality experience over the last decades shows a number of aspects affecting the shape of curves representing the mortality as a function of the attained age. Figures 7 and 8 illustrate the moving mortality scenario referring to the Italian male population, in terms of the graphs of survival tables and the curves of deaths respectively.

In particular, the following aspects can be singled out:

a) an increasing concentration of deaths around the mode (at old ages) of the curve of deaths is evident; so the graph of the survival table moves towards a rectangular shape, whence the term "rectangularization" to denote this feature (see Figure 9*(a)*);

b) the mode of the curve of deaths (which, owing to the rectangularization, tends to coincide with the maximum attainable age) moves towards very old ages; this aspect is called "expansion" of the survival function (see Figure 9*(b)*).

Other important features of the mortality trend are: an increase in life expectancy (at birth as well as at adult and old ages), a decrease in mortality rates (especially at adult and old ages).

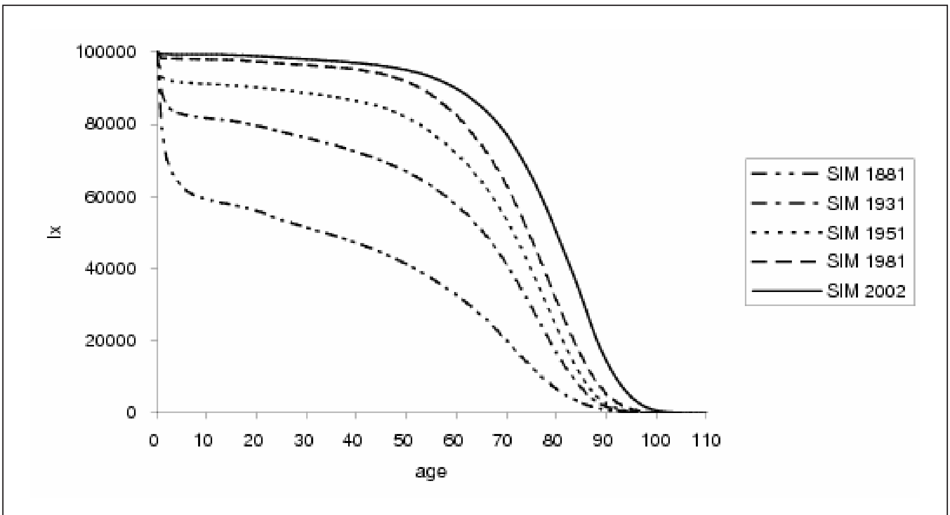

Fig. 7: Mortality trends in terms of survival tables.

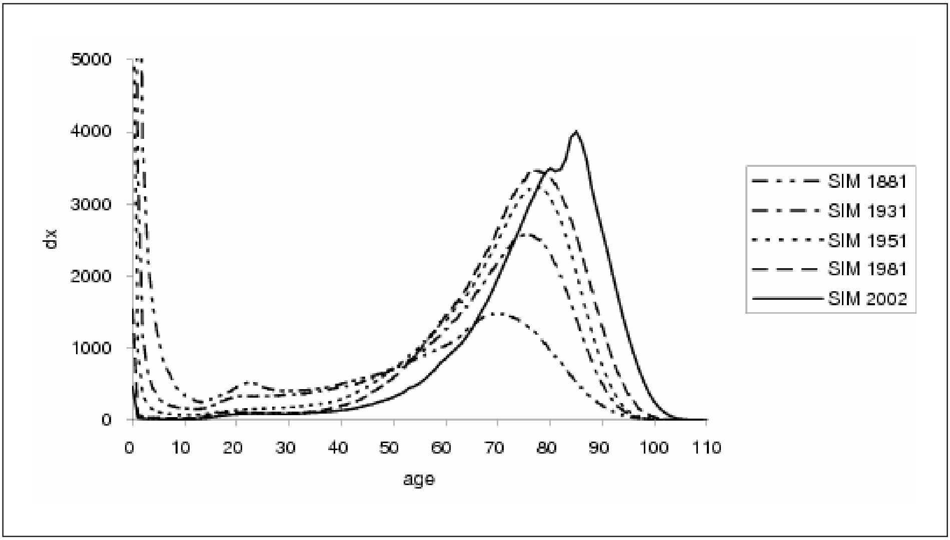

Fig. 8: Mortality trends in terms of curves of deaths.

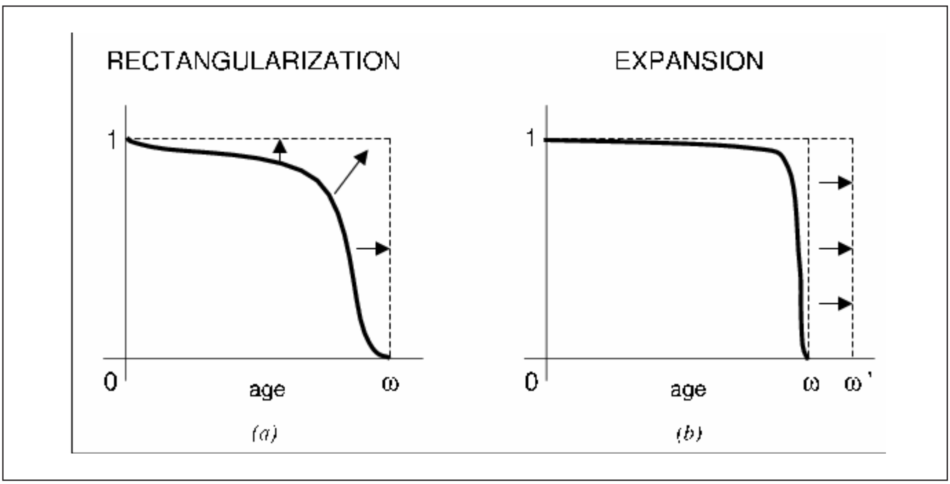

Fig. 9: Main aspects of mortality trends.

4.1 UNCERTAINTY IN FUTURE MORTALITY TRENDS

Observations on mortality suggest to adopt projected mortality tables (or laws) for the actuarial appraisal of life annuities (and other living benefits), i.e. to use mortality assumptions including a forecast of future mortality trends (for example, see Pitacco and Olivieri (2005) and the references therein). Notwithstanding, whatever hypothesis is assumed the future trend is random, whence an uncertainty risk arises. When this risk mainly refers to mortality trend at old ages, it is usually called “longevity risk”.

It is worthwhile to note that the experienced mortality trend and the uncertainty in future trend have a three-fold effect on the values of life annuities:

- decreasing probabilities of death imply a rise in the expected present value of a life annuity;
- the rectangularization phenomenon could lead to a reduction of riskiness conditional on a given survival function (e.g. in terms of conditional variance of the random present value of the annuity); however, from some statistical evidence it results that, when dealing with survival probabilities conditional on an adult-old age (as it is the case for immediate annuities), the rectangularization phenomenon does not necessarily occur, whence riskiness due to random fluctuations is not reduced;
- the expansion of the survival function implies a rise in unconditional riskiness (e.g. in terms of unconditional variance).

4.2 REPRESENTING THE LONGEVITY RISK

Because of uncertainty in future mortality trend, the probabilistic model used for representing mortality, in particular when dealing with life annuities, should allow for the assessment of longevity risk. This can be obtained in several ways. A simple and practicable approach, basically consisting of two steps, is described below (for details see Olivieri (2001)).

1) Choose a set of projected mortality tables (or survival functions, or forces of mortality, etc.) in order to express several alternative hypotheses about future mortality evolution. So, it is possible to perform a scenario testing, assessing the range of variation of quantities such as cash flows, profits, portfolio reserves, etc. This way, the sensitivities of these quantities to future mortality trend is investigated.
2) Assign a non-negative weight to each mortality hypothesis; the set of weights can be meant as a probability distribution on the space of hypotheses. Hence a probabilistic (or stochastic) approach can be adopted, calculating unconditional (i.e. non conditional on a particular hypothesis) variances, percentiles, etc., of the value of future cash flows, profits, etc.

Note that addressing just one given assumption on mortality, only the risk of random fluctuations is accounted for. So, a “deterministic” approach to longevity risk is adopted. On the contrary, performing the two steps above described both the risk of random fluctuations and the longevity risk are considered, whence a "probabilistic" approach is adopted. Clearly, in this context the terms "deterministic" and "probabilistic" only refer to the way in which the future mortality trend and the related uncertainty are accounted for.

Figure 10 illustrates the starting point of a probabilistic approach. A (finite) set of hypotheses is chosen, e.g. in terms of curves of deaths (or probability density functions of the random lifetime in an age-continuous setting), and a probability distribution over the set of hypotheses is assigned.

For comments about this (simple) approach to uncertainty in future mortality trend, the reader can refer to Olivieri and Pitacco (2006), where a more complex approach to representing longevity risk is also mentioned.

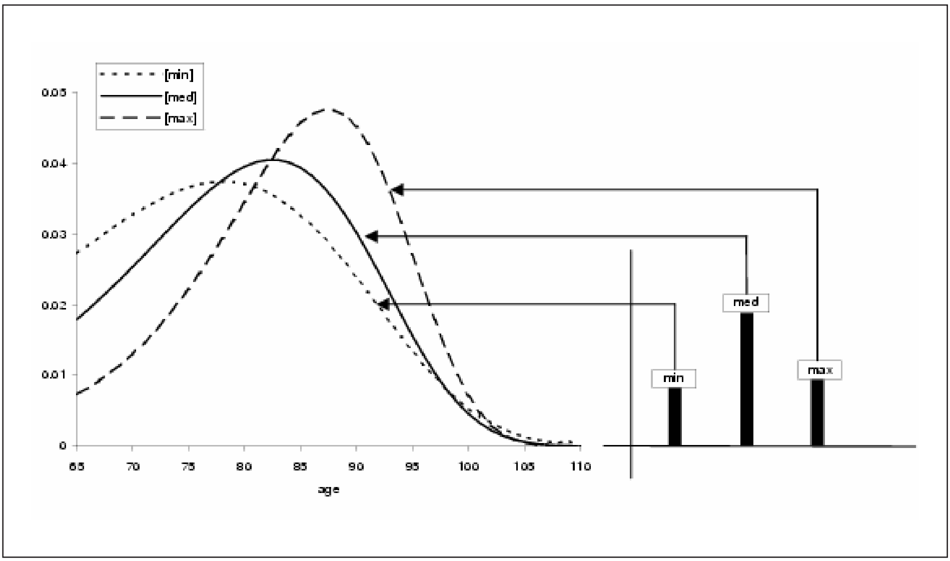

Fig. 10: Stochastic approach: a set of hypotheses about future mortality.

4.3 MEASURING THE LONGEVITY RISK

To assess the impact of longevity risk in a portfolio of life annuities, various "metrics" can be adopted, namely we can focus on several types of results (number of annuitants alive at various times, annual cash flows, discounted cash flows, level of the portfolio fund, etc.), and a number of single-figure indexes. In this Section, we first address a metrics based on the random present value of future outgoes for annuity payments. Secondly, we will focus on the random level of the portfolio fund and the shareholders' capital allocated to the portfolio, within a solvency framework. For further information on these issues, the reader can refer to Olivieri (2001), and Olivieri and Pitacco (2003, 2006).

As regards annuitants' mortality, we assume

$$\frac{q_x}{p_x} = G\,H^x \tag{4}$$

where, as usual, q_x and p_x respectively denote the mortality and the survival rate at age x, and the $\frac{q_x}{p_x}$'s are the so-called "odds". Note that the right-hand side of (4) is the third term in the well-known Heligman-Pollard law, i.e. the term describing the old-age pattern of mortality (see Heligman and Pollard (1980)). The parameter G expresses the level of senescent mortality and H the rate of increase of senescent mortality itself. The related survival function S(x) can be easily derived. A logistic shape of mortality rates q_x plotted against attained age x follows. Note that, due to mortality trends, parameters G and H should be cohort specific.

In order to represent mortality trends, we use projected survival functions. More precisely, we define three projected survival functions, denoted by $S^{[\min]}(x)$, $S^{[\text{med}]}(x)$ and $S^{[\max]}(x)$, expressing respectively a little, a medium and a high reduction in mortality with respect to period experience. Probabilities $\rho^{[\min]}$, $\rho^{[\text{med}]}$ and $\rho^{[\max]}$, are respectively assigned to the three survival functions. In the deterministic approach only $S^{[\text{med}]}(x)$ is used, whilst the probabilistic approach involves the three functions.

In what follows we refer to a portfolio consisting in one cohort of immediate single-premium life annuity contracts. Contracts are issued at time 0. We assume that all annuitants are aged x_0 at time t = 0. Their lifetimes are assumed to be independent of each other (conditional on any given survival function), and identically distributed. All annuities have a (constant) annual amount R. Expenses and related expense loadings are disregarded. N_0 denotes the (given) number of annuities at time t = 0.

First, consider the random present value at time 0 of the portfolio future payouts, $Y_0^{(\Pi)}$. The riskiness of the payout can be summarized by its variance or its standard deviation. A relative measure of riskiness is provided by the coefficient of variation, defined as the ratio of the standard deviation to the expected value. This relative measure of riskiness is often denoted, in actuarial mathematics, as the "risk index".

The risk index can be calculated conditional on a particular mortality survival function *S*, i.e. an assumption about future mortality scenario expressed by parameters *G*, *H*. In this case, only random fluctuations are accounted for.

$$\mathrm{CV}(Y_0^{(\Pi)} \mid S) = \frac{\sqrt{\mathrm{Var}(Y_0^{(\Pi)} \mid S)}}{\mathrm{E}(Y_0^{(\Pi)} \mid S)} \tag{5}$$

Conversely, the risk index can be calculated allowing for uncertainty in future mortality, weighting the scenarios with the relevant probabilities. In this case we have

$$\mathrm{CV}(Y_0^{(\Pi)}) = \frac{\sqrt{\mathrm{Var}(Y_0^{(\Pi)})}}{\mathrm{E}(Y_0^{(\Pi)})} \tag{6}$$

and both the random fluctuations and the systematic deviations are allowed for.

Turning to solvency issues, let Z_t denote the random portfolio fund (i.e. assets facing portfolio liabilities) at (future) time *t*, and $V_t^{(\Pi)}$ the random portfolio reserve set up at time *t*. The quantity M_t, defined as follows

$$M_t = Z_t - V_t^{(\Pi)} \tag{7}$$

represents the shareholders' capital at time *t*.

Solvency requirements are usually expressed in terms of M_t. For example, given a time horizon of T years, we say that the insurer has a solvency degree 1–ε if and only if

$$\Pr\left\{ \bigwedge_{t=1}^{T} M_t \geq 0 \right\} = 1 - \varepsilon \tag{8}$$

The capital required at time $t = 0$ is the amount $M_0^{(R)}$ such that condition (8) is fulfilled.

4.4 EXAMPLES

A possible choice for the parameters of the three Heligman-Pollard survival functions are shown in Table 7. As it emerges from such parameters, the projected functions have been obtained so that they perform the trends of expansion and rectangularization. Figures 11 and 12 show, respectively, the three survival functions and the related probability density functions. In any case, the maximum age has been set equal to 115 years; no uncertainty to this regard has been allowed, due to paucity of data.

Tab. 7: Mortality laws for annuitants.

	[min]	[med]	[max]
G	0.000042	0.000002	0.0000001
H	1.09803	1.13451	1.17215

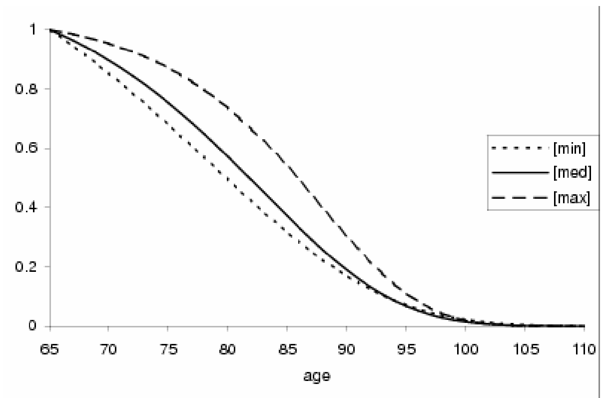

Fig. 11: Alternative scenarios: survival functions.

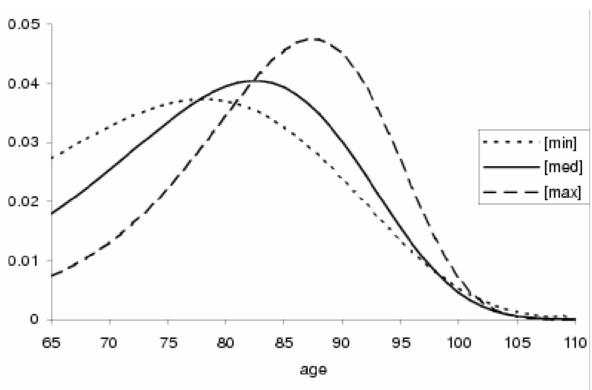

Fig. 12: Alternative scenarios: curves of deaths.

Table 8 provides a comparison between the coefficient of variation, as a function of the (initial) portfolio size N_0, in the deterministic and the stochastic setting. In the deterministic setting (conditional on the mortality scenario expressed by the projected survival function [med]), allowing for random fluctuations only, the pooling effect clearly emerges: actually the coefficient of variation tends to 0 as N_0 tends to ∞. Conversely, in the probabilistic setting, also accounting for systematic deviations, the coefficient of variation decreases as N_0 increases, but its limiting value is positive, showing the non-diversifiable part of the risk.

Tab. 8: Coefficient of variation: deterministic vs probabilistic approach.

N	$CV(Y_0^{(\Pi)} \mid [med])$	$CV(Y_0^{(\Pi)})$
1	31.98%	33.01%
10	10.11%	13.22%
100	3.20%	9.13%
1 000	1.01%	8.61%
10 000	0.32%	8.56%
100 000	0.10%	8.56%
…	…	…
∞	0%	8.56%

It is worth noting that the results above mentioned can be proved in analytical terms; to this purpose the reader can refer to Olivieri (2001), Olivieri and Pitacco (2006).

We now turn to the investigation of solvency issues. We address a portfolio of identical annuities, paid to annuitants of initial age $x_0 = 65$, with annual amount $R = 100$. As regards mortality assumptions, we adopt the Heligman-Pollard law, with parameters as described in Table 7.

The single premium (to be paid at entry) is calculated, for each policy, according to the survival function $S^{[med]}_{(x)}$ and with a constant annual interest rate $i = 0.03$. Further, we assume that for each policy in force at time t, $t = 0,1,\ldots$, a reserve must be set up, which is calculated according to such hypotheses.

In the deterministic approach to mortality, the probability distribution of the future lifetime of each insured is known, the only cause of uncertainty consisting in the time of death (see Section 4.2). The assessment of the solvency requirement is performed through simulation. In order to obtain results easier to interpret, we disregard profit; the actual life duration of the annuitants is thus simulated with the survival function $S^{[med]}_{(x)}$. Further, we assume that the yield from investments is equal to $i = 0.03$.

According to a probabilistic approach to uncertainty in future mortality, the assessment of the solvency requirement is obtained considering explicitly uncertainty in future mortality trends. To this aim, we consider the three survival functions $S^{[min]}(x)$, $S^{[med]}(x)$ and $S^{[max]}(x)$, weighted with the probabilities $\rho^{[min]}$, $\rho^{[med]}$ and $\rho^{[max]}$ representing the "degree of belief" of such functions.

The single premium for each policy and the individual reserve are still calculated with the survival function $S^{[med]}(x)$ and the interest rate i = 0.03. We still assume $\rho^{[min]} = 0.2$, $\rho^{[med]} = 0.6$, $\rho^{[max]} = 0.2$ (reflecting the fact that $S^{[med]}(x)$, which is used for pricing and reserving, is supposed to provide the most reliable mortality description).

Obviously, the investigation is carried out via simulation. We now deal with two causes of uncertainty: the actual distribution of the future lifetimes and the time of death of each insured. First, the survival function must be chosen (through simulation) and then, assuming that under a given lifetime distribution the annuitants are independent risks, the actual duration of life of each person is simulated.

To illustrate the results, we consider the quantity

$$\mathrm{QM}^{[\cdot]}(N_0) = \frac{M_0^{(\mathrm{R})}}{V_0^{(\Pi)}} \tag{9}$$

In Figure 13 solvency requirements are shown in terms of the ratios (9), calculated according to the deterministic ($\mathrm{QM}^{[det]}(N_0)$) and the probabilistic ($\mathrm{QM}^{[prob]}(N_0)$) approach respectively, and plotted against the (initial) portfolio size N_0. A ruin probability $\varepsilon = 0.025$ and a time horizon of T 110 – 65 = 45 years (assuming 110 as the maximum age) have been chosen. According to the deterministic approach only the random fluctuations are accounted for, whence the solvency requirement tends to 0 as the (initial) portfolio size N_0 diverges, thanks to the pooling effect. Conversely, the probabilistic approach allows for systematic deviations also, and hence the solvency requirement keeps high even for large portfolio sizes.

A deterministic approach to solvency requirements could be used (and actually is sometime used) accounting for uncertainty in future mortality trends (i.e. longevity risk) also. Let $V_0^{(\Pi)[W]}$ denote the (initial) reserve calculated according to a "worst case" basis, i.e. allowing for a very strong mortality improvement, and $V_0^{(\Pi)[B]}$ the (initial) reserve according to a "bad case" basis, i.e. a strong mortality improvement. Clearly

$$V_0^{(\Pi)} < V_0^{(\Pi)[B]} < V_0^{(\Pi)[W]} \tag{10}$$

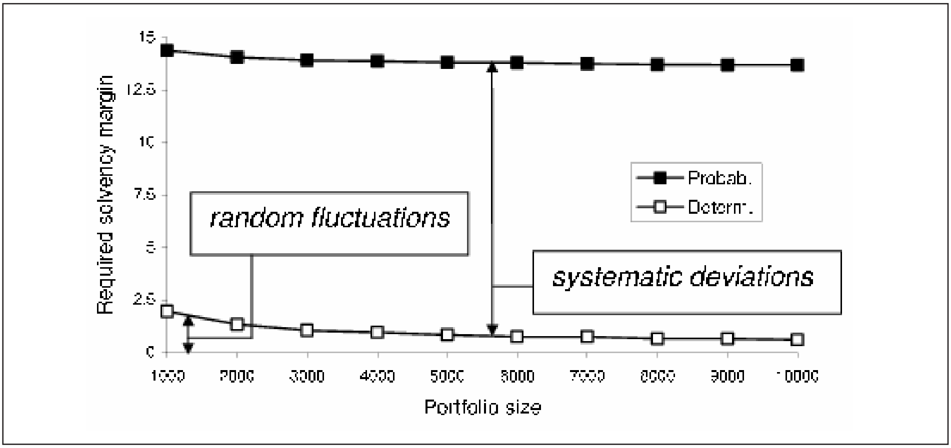

Fig. 13: Solvency requirements for mortality risks {solv1}.

Let $QV^{[\bullet]}$ denote the ratio defined as follows

$$QV^{[\cdot]} = \frac{V_0^{(\Pi)[\cdot]}}{V_0^{(\Pi)}} - 1 \tag{11}$$

Obviously ratios $QV^{[\bullet]}$ are independent of both the portfolio size N_0 and the probability ε. From (10) it follows that

$$0 < QV_0^{[B]} < QV_0^{[W]} \tag{12}$$

Conversely, in a probabilistic framework we find

$$QM_0^{[det]}(N_0) < QM_0^{[prob]}(N_0) \tag{13}$$

(see, for example, Figure 13). Comparing ratios $QV^{[\bullet]}$ and $QM^{[\bullet]}(N_0)$does not lead to general conclusions. However, a likely situation is represented by Figure 14. The following aspects should be noted.

1) Allocating shareholders' capital in the measure suggested by the "worst case" reserve leads to a huge and likely useless capital allocation, whatever the portfolio size N_0 may be; see the value of $QV_0^{[W]}$ compared to $QM^{[prob]}(N_0)$.
2) A "bad case" reserve based capital allocation can result in a poor capability of facing the mortality risks when small portfolios are concerned; see the portfolio sizes such that $QV_0^{[B]} < QM^{[prob]}(N_0)$. Conversely, a too high capital allocation occurs for larger portfolios; see the interval where $QV_0^{[B]} < QM^{[prob]}(N_0)$.

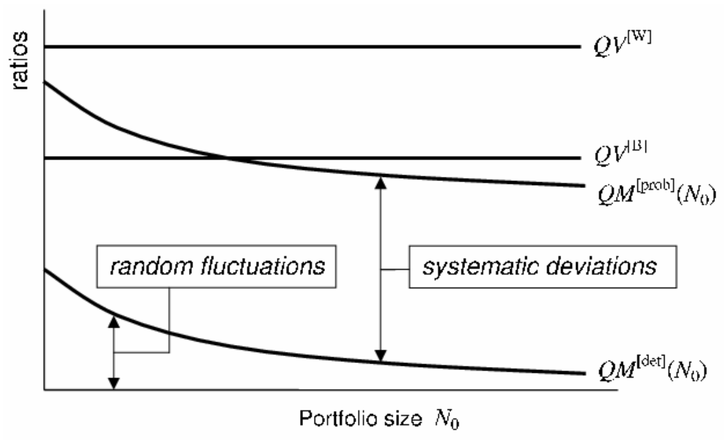

Fig. 14: Reserving and solvency requirements {ratios}.

Thus, setting aside a solvency margin simply based on the comparison of reserves calculated with different survival functions (as some practice suggests) on the one hand would disregard the risk of random fluctuations (which obviously can be considered separately) and on the other would disregard a valuation of the probability of ruin, possibly leading to not sound capital allocation.

5. CONCLUDING REMARKS

A great attention is currently devoted to the assessment of the risk profile of an insurance company. New issues concerning insurers' solvency contribute to increase the interest in appropriate tools for evaluating the impact of the various sources of risk on portfolio results (e.g. cash flows, profits, reserves, etc.).

Among the risks which affect life insurance and annuity portfolios, both investment risks and mortality risks deserve careful analyses and require the adoption of proper management solutions.

Literature on investment risks is very rich. Several tools have been proposed in this respect, and implemented in practice as well. Conversely, the analysis of mortality risks still requires investigations, especially as far as the longevity risk is concerned.

When assessing the risk profile of a life insurer, riskiness arising from the behavior of mortality should be analyzed via appropriate tools. In particular, the traditional deterministic approach to mortality modeling should be rejected and replaced by a stochastic approach allowing for process risk and uncertainty risk as well.

In this paper two examples of stochastic mortality modeling have been presented and discussed, concerning death benefits and life annuities respectively. While the latter focuses on the well known problem of longevity risk, the former deals with the risk of random fluctuations in a portfolio of insurance products with a positive sum at risk.

Although in practice the importance of mortality risks is often underestimated when dealing, for example, with endowments or term assurances, and a deterministic approach to mortality is consequently adopted, it should be stressed that an appropriate assessment of the insurer's risk profile should account for all types of risks, not disregarding, in particular, the effectiveness of the pooling effect when small portfolios are addressed.

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MORTALITÀ STOCASTICA NELLE ASSICURAZIONI VITA

Riassunto

La quantificazione del profilo di rischio di un assicuratore (o di un fondo pensioni) richiede un'esplicita considerazione dell'aleatorietà dei numeri di decessi in una data popolazione, e dunque un approccio stocastico alla rappresentazione della mortalità. Questo lavoro raccoglie alcune considerazioni sui modelli impiegabili per rappresentare il rischio di "scarti accidentali" (o "process risk") ed il rischio di "scarti sistematici" ("uncertainty risk", con particolare riguardo al longevity risk), nonché per quantificare gli effetti di tali rischi sui risultati tecnici.