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A NOTE ON THE EXISTENCE AND UNIQUENESS OF THE NEYMAN–PEARSON TEST

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Abstract

This note discusses the problem of the existence and uniqueness of the MP test. The necessary and sufficient conditions for the existence and uniqueness of the test are established.

Key words: factored function, range dependent on θ, randomised and non randomised MP test.

Textbooks on testing statistical hypotheses devote little attention to the problem of the existence and uniqueness of the test based on the Neyman–Pearson fundamental Lemma. E. L. Lehmann (1986) himself states that it "*is most powerful (MP) at level α... unless there exists a test of size < α and with power 1*" (p.74) and that it "*... is uniquely determined except on the set on which $p_1(x) = kp_0(x)$*" (p. 76), but he does not specify under which conditions this occurs.

The following note examines the problems of existence and uniqueness of the Neyman–Pearson test, specifying these conditions for each.

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be a random sample with range $\mathcal{X}$ and probability density function (pdf) $\varphi(\boldsymbol{x};\theta)$, where the parameter $\theta \in \Re$ is unknown.

The MP test of size α for testing H_0: $\theta = \theta_0$ versus H_1: $\theta = \theta_1$ is based on the Neyman–Pearson Lemma and is given by:

$$\tau(x)=\begin{cases}1 & \text{if } \varphi(x;\theta_1)>k\varphi(x;\theta_0)\\ \gamma & \text{if } \varphi(x;\theta_1)=k\varphi(x;\theta_0)\\ 0 & \text{if } \varphi(x;\theta_1)<k\varphi(x;\theta_0)\end{cases} \tag{1}$$

where γ is such that:

$$E_{\theta_0}(\tau(X))=\alpha \tag{2}$$

In order to specify the conditions for the existence and uniqueness of the test (1) with size (2), let $\mathcal{X}_0$ represent the range of X under H_0 and similarly $\mathcal{X}_1$ the range of X under H_1 and let E_1, E_2 and E_3 be: $E_1 = \mathcal{X}_0 - \mathcal{X}_1$, $E_2 = \mathcal{X}_1 - \mathcal{X}_0$, and $E_3 = \mathcal{X}_0 \cap \mathcal{X}_1$.

Clearly, the problem of testing hypotheses is not trivial if E_3 has non-null measure. That is the assumption we make below.

Lemma 1 *A test with power 1 and size $\alpha^* < 1$ exists iff* E_1 *has non-null measure.*

The proof is straightforward. Once we see that a test has power 1 iff its critical region coincides almost everywhere with $\mathcal{X}_1$, and that this test has size:

$$\alpha^* = E_{\theta_0}(\tau(X)) = P_{\theta_0}(X \in \mathcal{X}_1) = 1 - P_{\theta_0}(X \in \mathbf{E}_1)$$

the statement follows.

Remark 1 A test with power 1 and size $\alpha^* < 1$ exists only if $\mathcal{X}$ depends on θ, otherwise $\mathcal{X}_0 = \mathcal{X}_1$ and therefore $E_1 = \varnothing$.

Remark 2 If the test of power 1 and size $\alpha^* < 1$ exists, then the MP test (1) exists only for size $\alpha < \alpha^*$.

The following Lemma is a variation on a similar result obtained by Migliorati (1999).

Lemma 2 *The ratio $\lambda(x) = \varphi(x;\theta_1)/\varphi(x;\theta_0)$ is constant on the set $A \subseteq E_3$ of non–null measure iff the pdf can be factored on A as follows:*

$$\varphi(x;\theta) = f(x)h(\theta) \qquad \text{for all } x \in A \tag{3}$$

Proof *Necessary condition.* If $\lambda(x) = k$ on A, then, on A, k depends only on θ_0 and θ_1 but not on x; i.e.:

$$\frac{\varphi(x;\theta_1)}{\varphi(x;\theta_0)} = \mathrm{k}(\theta_1,\theta_0) \qquad \text{for all } x \in \mathrm{A}$$

and the two pdf's differ only in a multiplicative constant dependent on θ. It follows that:

$$\varphi(\boldsymbol{x};\theta) = f(\boldsymbol{x})h(\theta) \qquad \text{for all } \boldsymbol{x} \in A$$

and so $\varphi(\boldsymbol{x};\theta)$ can be factored as in (3) above.

Sufficient condition. If $\varphi(\boldsymbol{x};\theta)$ can be factored on A as in (3), the ratio $\lambda(\boldsymbol{x})$ becomes:

$$\lambda(\boldsymbol{x}) = \frac{f(\boldsymbol{x})h(\theta_1)}{f(\boldsymbol{x})h(\theta_0)} = \frac{h(\theta_1)}{h(\theta_0)} \qquad \text{for all } \boldsymbol{x} \in A$$

and remains constant on A, as previously stated.

Remark 3 The pdf $\varphi(\boldsymbol{x};\theta)$ can be factored on $\mathcal{X}$ only if $\mathcal{X}$ depends on θ, indeed it follows from (3) that:

$$\int_{\mathcal{X}} f(\boldsymbol{x})d\boldsymbol{x} = \frac{1}{h(\theta)}$$

so $\mathcal{X}$ depends on θ.

Remark 4 Iff $\varphi(\boldsymbol{x};\theta)$ can be factored, then from some sizes α $(0 \le \alpha \le 1)$, together with the MP randomised test (1), there exist infinite non-randomised tests with the same size and power as test (1).

From all of the above, we can make the following statement.

Conclusion The MP test (1) based on the Neyman-Pearson fundamental Lemma may not exist or not be unique only if $\mathcal{X}$ depends on θ and, more specifically:

a) test (1) does not exist iff E_1 has non-null measure and $\alpha > \alpha*$, where $\alpha*$ is the size of the test with power 1;
b) test (1), where it exists, is not unique, because there are infinite non–randomised tests with the same size and power as test (1), iff pdf $\varphi(\boldsymbol{x};\theta)$ can be factored in the form (3).

Example Let X be the random variable with pdf:

$$\varphi(x;\theta) = \exp(-x+\theta) \qquad x \ge \theta \tag{4}$$

where θ is unknown. We wish to test the hypothesis H_0: $\theta = 0$ versus H_0: $\theta = 1$ on the basis of a single observation x.

Here, $\mathcal{X}$ depends on θ and the ratio $\lambda(x)$ is:

$$\lambda(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ e & \text{if } x \ge 1 \end{cases}$$

Since $E_1 = [0,1)$ is a set of non–null measure, the test with power 1 and size $\alpha* < 1$ exists. This test is given by:

$$\tau(x)=\begin{cases}1 & \text{if } x\geq 1\\ 0 & \text{if } 0\leq x<1\end{cases}$$

and its size is:

$$\alpha^* = E_{\theta_0}\left(\tau(X)\right)=\int_1^{\infty}\exp(-x)dx=\exp(-1)=0.36<1.$$

It follows that:

a) for $\alpha > \alpha^* = 0.36$, test (1) does not exist:

b) for $\alpha \leq \alpha^* = 0.36$, test (1) exists and is given by:

$$\tau(x)=\begin{cases}\gamma & \text{if } x\geq 1\\ 0 & \text{if } 0\leq x<1\end{cases}$$

where $\gamma = \alpha e$ and has power:

$$1-\beta = E_{\theta_1}\left(\tau(X)\right)=\gamma\int_1^{\infty}\exp(-x+1)dx=\gamma. \tag{5}$$

However, since the pdf $\varphi(x;\theta)$ can be so factored on $A = E_3 = [1, \infty)$:

$$\varphi(x;\theta)=\exp(-x)\exp(\theta)$$

there exist infinite tests with the same size and power as test (1) and with the following form:

$$\tau(x)=\begin{cases}1 & x\in B\\ 0 & x\in\left\{[0,\infty)-B\right\}\end{cases}$$

where B is any subset of $[0, \infty]$ such that:

$$E_{\theta_0}\left(\tau(X)\right)=\int_B\exp(-x)dx=\alpha \qquad \alpha\leq 0.36.$$

These tests have power:

$$1-\beta = E_{\theta_1}\left(\tau(X)\right)=\int_B\exp(-x+1)dx=e\int_B\exp(-x)dx=e\alpha=\gamma$$

which coincides with (5).

The above holds *mutatis mutandis* for discrete random variables.

ACKNOWLEDGEMENTS

I would like to thank the anonymous referee for useful suggestions.

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SULL'ESISTENZA E UNICITÀ DEL TEST DI NEYMAN–PEARSON

Riassunto

In queste brevi note vengono affrontati i problemi di esistenza e unicità del test NP. In particolare si evidenziano le condizioni necessarie e sufficienti per l'esistenza e l'unicità del test.