PROBABILITY AND INFERENCE IN THE WORKS OF CORRADO GINI: SOME REMARKS

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Abstract *The early papers of C. Gini in the years 1906-1911 were mainly devoted to statistical applications of probability. Gini uses probabilities either when they ensue from game structures, or when they are given by relative frequencies in large samples. In his 1911 paper, Gini exposes two procedures in order to obtain posterior probabilities. Both procedures – with applications to demographic phenomena – are commented in detail in this paper. In the final section, Gini's approach to statistical inference is briefly resumed; as is well known, Gini' approach to statistical inference was strictly Bayesian, and based on an objective prior distribution. We devote attention also to two special problems in the theory of hypotheses testing and statistical estimation: the comparison of Bayesian and confidence intervals, and the problem of dealing with point null hypotheses.*

Key words: *Bayesian inference, Empirical Bayes, Posterior probabilities, Statistical inference, Point null hypotheses.*

1. EARLY RESEARCH OF GINI ON PROBABILITY

Corrado Gini is chiefly known – especially outside Italy – for his research and proposals about descriptive statistics often applied to income and wealth. Among others, the paper by Giorgi and Gubbiotti (2016) reports a survey of his studies. However, from the beginning of his research work in 1906 and repeatedly later he was interested in dealing with the foundations of statistical inference and its applications. Gini writes, at the beginning of one of his latest papers (1964): "On revient toujours aux premières amours … I must confess that my first love was probability theory".

Some of the original papers, all written in Italian before 1912, are not easily available, especially abroad; fortunately enough, they can be found in the volumes of Gini papers (Gini 1968, two volumes, and 2001), issued by the Istituto di Statistica e Ricerca Sociale «C. Gini» of Rome University, and by the Statistics Department of Bologna University. The reference papers pertain to the period 1906-1908 (Gini, 1906, 1907, 1908a, 1908b). The object of these papers was the concept of probability, its determination in particular cases and its statistical applications (mostly to demographic phenomena).

In the 1906 paper, Gini displays a very wide and deep examination of the contributions on probability made by many important scholars since the middle of the seventeenth century, with a careful distinction of the studies about the concept of probability (also called *logical* theory of probability), and about the determination – or measure – of probabilities in particular cases (also called *psychological* theory of probability). In doing so, Gini first examines the concept of *chance*. Although avowing, and stating beforehand, his uncompromised support to the causal structure of the world ("any event is subjected to the law of causality", Gini 1906, p. 21), he expounds several examples for which the foresight of an event cannot be established with certainty. He is especially interested in the chance events whose relative frequency tends (although irregularly) to a constant p $(0 < p < 1)$, namely the probability of the event. Gini admits that a phenomenon can be classified as a chance phenomenon also when its probability varies according to a given law (Gini 1906, p. 23).

For the empirical assessment of probabilities, Gini distinguishes the cases: (a) when we assume a given prior distribution for the possible outcomes, in which case all probabilities are deductively derived (as in the early investigations of Pascal, Fermat, de Moivre, James Bernoulli, about games of chance), and (b) when the prior distribution is lacking or incomplete, the determination of probabilities lies – at least partially – on experience and observation (inductive method).

Here lies the fundamental distinction between mathematical research about random phenomena and statistical research based on real observations. Gini actually employs frequencies based on large samples as good approximations of probabilities, especially when derived from demographic data.

Gini's conception of probability is essentially objective. In the earliest applications on demographic phenomena, he estimated probabilities through the relative frequencies computed on very large data sets. In the cases of games (or trials, or experiments) of chance with equally likely outcomes, such as drawing balls from a bowl or urn, he assumed for the model probabilities the *classical definition*, i.e. fraction k / n, being n the number of all equally likely outcomes and k the number of the outcomes complying with a given property or condition. Gini stresses that frequency k / n does not pertain to the conceptual side of probability, but only to its determination, thus avoiding any vicious circle.

Concerning applications, Gini stressed that: (a) all determinations of probabilities are to be considered as approximate values of the true probabilities, and (b) every probability must be referred and associated to a particular sampling space, often a subset of a reference sampling space that is relevant for the application at hand (Gini, 1907, 1908a; Fisher, 1937; Frosini, 1999).

Gini states that we can speak of probability only for phenomena that are susceptible of repetition (Gini, 1906, p. 99); however, he admits that a probability can be assigned to a single (individual) occurrence of such a phenomenon when this occurrence can be considered as a generic event yielded by the collective phenomenon. In this respect, Gini declares that Bayes' rule (for which a gratuitous – although mathematically convenient – prior probability is assumed) is devoid of any scientific value (Gini, 1906, p. 104). However, he acknowledges that one can rely on final estimates if the dataset is very large, for, in this case, the computation of the final probabilities is practically independent of the prior hypothesis adopted.

The early research of Gini on demographic phenomena mirrors the above viewpoint on probability. In the volume 'Il sesso dal punto di vista statistico' (Gini, 1908a) and in the paper 'Considerazioni sulle probabilità a posteriori – Applicazioni al rapporto dei sessi nelle nascite umane' (Gini, 1911) Gini makes a large use of models and research about probability statements. From the interview with A.W.F. Edwards (Palla, 2015), we realise that Ronald Fisher had a copy of Gini's volume, and discussed some of its content with Edwards, who started his first research work on topics first studied by Gini.

2. POSTERIOR PROBABILITIES IN GINI'S PAPER OF 1911[1]

Gini's paper of 1911 titled "Considerazioni sulle probabilità a posteriori – Applicazioni al rapporto dei sessi nelle nascite umane" deserves a special attention; the first part of this paper has been resumed in the volume "Statistica e Induzione" (Gini, 2001). In this paper, Gini suggests two different approaches in order to derive posterior probabilities of demographic (and also of more general) interest.

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[1] This section is taken from the author's section with same title, which appeared in the online proceedings of the meeting "Statistics and Demography: The Legacy of Corrado Gini" edited by Corrado Crocetta, CLEUP, 2015.

Unfortunately, the symbolism used by Gini, although adequate and always clearly defined, is rather awkward, and its meaning differs in the two approaches. With reference to the symbolism first introduced in his paper, we write a conditional probability as follows:

$$ {}_{m,n}P_{x,s} = \Pr(X_s = x \mid X_n = m) \qquad (1) $$

meaning that this is the probability that an event A occurs x times in s trials, upon the condition that the same event has already occurred m times in n previous trials.

Following Gini's first approach, we assume that the event A may occur under one of v systems of causes; let $Y = y$ mean that y is the system of causes actually working when the previous n trials were realised. Gini calls

$$ {}_{m,n}\, r_y = \Pr(Y = y \mid X_n = m) $$

the probability that, given m successes (event A) in the previous n trials, the system of causes concretely operating was y. Hence the probability (1) may be written as a total probability:

$$ \Pr(X_s = x \mid X_n = m) \;=\; \sum_{y=1}^{v} \Pr(Y = y \mid X_n = m) \Pr(X_s = x \mid Y = y). $$

By Bayes' theorem

$$ \Pr(Y = y \mid X_n = m) = \frac{\Pr(Y = y)\Pr(X_n = m \mid Y = y)}{\Pr(X_n = m)}, \qquad (2) $$

where

$$ \Pr(X_n = m) = \sum_{y=1}^{v} \Pr(Y = y)\Pr(X_n = m \mid Y = y) $$

and also

$$ \Pr(X_s = x \mid X_n = m) = \sum_{y=1}^{v} \frac{\Pr(Y = y)\Pr(X_n = m \mid Y = y)}{\Pr(X_n = m)} \Pr(X_s = x \mid Y = y) \qquad (3) $$

Gini observes that "the practical usefulness of this result is almost null", because we are never acquainted about the probability $\Pr(Y = y)$ that the system of causes y operates in a number of trials and so we usually do not know the probability

$$ {}^{y}P = \Pr(A \mid Y = y). $$

Gini is aware that some scholars made the above probabilities operational by assuming "that v be infinitely large, that p_y is equal for all systems of causes y, and that ${}^yP$ can assume all real values between 0 and 1". Gini opines that "*these hypotheses are wholly arbitrary*".

These hypotheses, or assumptions, and corresponding results, have been put forward – following the masterpiece of Thomas Bayes – by several authors. Among them, Laplace (1820), De Morgan (1838), Venn (1866, 1876) and Fisher (1959). Gini quotation is from Czuber (1902). The general result coming from the above assumptions is as follows:

$$\Pr(X_s = x \mid X_n = m) = \frac{s!(m+x)!(n-m+s-x)!(n+1)!}{x!(s-x)!(s+n+1)!m!(n-m)!}. \qquad (4)$$

From expression (4) we obtain:

$$_{m,n}P = \Pr(A \mid X_n = m) = \frac{m+1}{n+2}. \qquad (5)$$

This result is simple and extensively discussed in the literature. From Fisher (1959, p. 24), by assuming as before a prior distribution for p uniform between 0 and 1, the posterior density for p is proportional w.r.t. the likelihood, and is precisely, after a successes have been obtained in $a + b$ independent trials (Frosini, 2009, p. 401):

$$f(p) = \frac{(a+b+1)!}{a!b!} p^a (1-p)^b.$$

The probability of success for a new trial becomes

$$\int_0^1 p f(p) dp = \frac{a+1}{a+b+2}. \qquad (6)$$

(this is also called "rule of succession"); when $a + b = n$ and $a = m$, we find again formula (5).

Gini follows quite a different approach to determine the posterior (or, more precisely, conditional) probabilities, without recurring to prior probabilities. What is relevant in this approach are the so called *direct results*, which are the possible outcomes of event A in $n + s$ trials, given that it occurred m times in n trials. Thus the possible direct results are given by m, m+1, ... , m+s (s+1 direct results altogether). We may call $X_{n+s} = m + i$ the i-th direct result (where X_{n+s} is the number of successes in n+s trials, and $f_i = \Pr(X_{n+s} = m + i)$ is its probability).

Under the independence assumption we can write:

$$\Pr(X_n = m, X_{n+s} = m+i) = \Pr(X_n = m) \Pr(X_{n+s} = m+i \mid X_n = m).$$

This probability is also equal to

$$\Pr(X_{n+s} = m+i) \Pr(X_n = m \mid X_{n+s} = m+i)$$

and hence we can write

$$\Pr(X_{n+s} = m+i \mid X_n = m) = \frac{f_i \Pr(X_n = m \mid X_{n+s} = m+i)}{\Pr(X_n = m)} . \tag{7}$$

Given that

$$\Pr(X_n = m) = \sum_{i=0}^{s} \Pr(X_{n+s} = m+i) \Pr(X_n = m \mid X_{n+s} = m+i)$$

we can write

$$\Pr(X_{n+s} = m+x \mid X_n = m) = \frac{f_x \Pr(X_n = m \mid X_{n+s} = m+x)}{\sum_{i=0}^{s} f_i \Pr(X_n = m \mid X_{n+s} = m+i)} . \tag{8}$$

By using binomial probabilities, with probability $P = \Pr(A)$ constant over all trials and independent between the first n observations and the successive s ones, one obtains the following formula for the probability of $X_n = m$ under the condition that $X_{n+s} = m + x$:

$$\Pr(X_n = m \mid X_{n+s} = m + x) = \frac{s!(m+x)!(n+s-m-x)!n!}{x!(s-x)!(n+s)!m!(n-m)!} . \tag{9}$$

This expression may be replaced in the numerator of (7), thus obtaining an operational formula for ${}_{m,n}P_{x,s} = \Pr(X_{(s)} = x)$:

$$\Pr(X_{(s)} = x) = \frac{f_x \dfrac{(m+x)!(n+s-m-x)!}{x!(s-x)!}}{\sum_{i=0}^{s} f_i \dfrac{(m+i)!(n+s-m-i)!}{i!(s-i)!}} . \tag{10}$$

Gini dwells on the special and important case when $s = 1$ (i.e. by conditioning on $(n + 1)$ trials). The results are as follows:

$${}_{m,n+1}P_{0,1} = \Pr(X_{n+1} = m \mid X_n = m) = \frac{n-m+1}{n+1}$$

if the $(n+1)$-th outcome is a failure, and

$$_{m+1,n+1}P_{1,1} = \Pr(X_{n+1} = m + 1 \mid X_n = m+ 1) = \frac{m+1}{n+1}$$

if the (n+1)-th outcome is a success. The sum of the two probabilities is (n+2)/(n+1) > 1. In fact, the two events are not complementary (there is a change in the conditioning event).

By applying a total probability to the above case one gets:

$$\Pr(X_n = m) = \Pr(X_{n+1} = m+1) \Pr(X_n = m \mid X_{n+1} = m+1) + \\ + \Pr(X_{n+1} = m) \Pr(X_n = m \mid X_{n+1} = m)$$

$$= \Pr(X_{n+1} = m+1)\,(m+1)/(n+1) + \Pr(X_{n+1} = m)\,(n-m+1)/(n+1).$$

By applying the Bayes' formula we get:

$$_{m,n}P = \Pr(A \mid X_n = m) = \Pr(X_{n+1} = m+1 \mid X_n = m) = \\ = \frac{R(m+1)}{R(m+1)+n-m+1} \qquad (11)$$

where $R = f_1/f_0 = \Pr(X_{n+1} = m+1)/\Pr(X_{n+1} = m)$.

After some interesting examples, Gini introduces some assumptions that raise questions not easy to be answered. Remembering that $f_i = \Pr(X_{n+s} = m + i) = \Pr(X_{(s)} = i)$ is the probability of the i-th direct result, the first basic assumption made by Gini is that all these probabilities *are equal*; this assumption entails a drastic simplification in formula (8), which becomes

$$\Pr(X_{n+s} = m+x \mid X_n = m) = \frac{\Pr(X_n = m \mid X_{n+s} = m+x)}{\sum_{i=0}^{s} \Pr(X_n = m \mid X_{n+s} = m+i)} . \qquad (12)$$

Nonetheless, the above equality of all probabilities f_i is inconsistent with the derivation of formula (9), which assumes the validity of binomial probabilities in the case of c trials:

$$\Pr(X_c = b) = \frac{c!}{b!(c-b)!} P^b (1-P)^{c-b}$$

typically for $0<P<1$.

The equality between all successive probabilities f_i can only be ensured if $\Pr(A) = 0$ or 1; moreover, concerning formula (11) and $0<P<1$, only when $P = 1/2$ (success and failure have the same probability). A like result is obtained by Gini also under the assumption of $P \sim U(0,1)$ (P uniformly distributed between 0 and 1). Gini quotes a result, ascribed to Luigi Galvani, which allows an important simplification of formula (12):

$$\sum_{i=0}^{s} \Pr(X_n = m \mid X_{n+s} = m+i) = \frac{n+s+1}{n+1} \qquad (13)$$

which becomes

$$\Pr(X_{n+s} = m+x \mid X_n = m) = \frac{s!(m+x)!(n-m+s-x)!(n+1)!}{x!(s-x)!(s+n+1)!m!(n-m)!} . \qquad (14)$$

First of all, note that (13) is > 1 for $s \geq 1$; thus it is not a probability, while the denominator of (8) is the probability $\Pr(X_n = m)$; in other words, formula (12) looks like a Bayes's formula but cannot be obtained through it.

As a special case, when $s = 1$ and $x = 1$, formula (14) complies with the reasonable result

$$\Pr(X_{n+1} = m+1 \mid X_n = m) = \frac{m+1}{n+2} \qquad (15)$$

thus *finding anew the old result* (5) (Laplace-De Morgan-Venn-Fisher) obtained through a correct Bayes' formula, under the hypothesis that $P = \Pr(A)$ is a random variable uniform between 0 and 1.

This equality of results looks rather strange and seems to support – in Gini's view – the correctness of his approach. Nevertheless, the above equality formula(5) = formula(15) is obtained without any theoretical support, and does not seem to imply any special meaning (contrary to the interpretation by Gini).

Besides, *assuming a uniform distribution* for all the probabilities of success $0<P<1$, Gini obtains for the probability of (m+1) successes in (n+1) trials, and also for m successes in (n+1) trials, the same probability:

$$\Pr(X_{n+1} = m+1) = \frac{(n+1)!}{(m+1)!(n-m)!}\int_0^1 P^{m+1}(1-P)^{n-m}dP = \frac{1}{n+2}$$

$$\Pr(X_{n+1} = m) = \frac{(n+1)!}{m!(n-m+1)!}\int_0^1 P^{m}(1-P)^{n-m+1}dP = \frac{1}{n+2} .$$

Also this equality looks rather strange, and we are led to resume Gini's first comment that *these hypotheses are wholly arbitrary*.

Gini observes that sometimes we only know that the event *A* has occurred (m+y) times in (n+t) trials, where $t > s$ (in case we are interested in the occurrence of *A* in *s* additional trials after the first *n*). Gini calls this kind of observation an *indirect result*. Hence he generalizes the approach that exploits the *direct results* in several similar formulas.

Just to work out (it is our interpretation) the ensuing *mathematical results*, also in this case Gini introduces the assumption (commented at length above) that the probabilities

$$f_y = \Pr(X_{n+t} = m+y) \tag{16}$$

are constant for every $y = 0, 1, \dots, t$. Again, the results obtained under this assumption seem wholly unacceptable, as the same assumption is untenable. Gini presents some applications of the above theory by using the general results and not the results derived under assumption (16).

It is worth pointing out that Gini (1911) is the first work in literature that employs tools and purposes within the inferential approach nowadays called *Empirical Bayes*. In the words of Chiandotto (1978, p. 266), “Gini suggests a procedure aimed at estimating prior probabilities (smooth empirical Bayes) and two alternative procedures (method of direct results and method of indirect results) that allow estimating posterior probabilities *without* passing through a previous estimation of prior ones (simple empirical Bayes)”.

A generalisation and an arrangement of the above Gini’s procedures were made by Pompilj (1951), while Forcina (1987), after resuming the above comments within a general Empirical Bayes framework, observed that the ‘anticipation’ made by Gini as early as 1911 “is now widely recognized”.

3. THE APPROACH OF C. GINI TO STATISTICAL INFERENCE

The ideas of Corrado Gini about statistical inference are best explained in the papers of 1939 and 1943, presented before the Italian Statistical Society (SIS: Società Italiana di Statistica). The approach to statistical inference propounded by Gini is well known in the literature. Among the many commentaries including the above papers we may quote those by Herzel and Leti (1978), Forcina (1982, 1987), Frosini (1989), Forcina and Giorgi (2005).

For a summary of Gini’s approach, we will refer to Frosini (1989, 2005); the latter paper is mostly devoted to Gini’s treatment of statistical intervals used in estimation, though this treatment is necessarily made consistent with the general approach to inference.

In order to mimic the process for realising a test of significance, we calculate the probability π_a that some value x_0 (or greater than x_0) be *accidental* under a given distribution H_0 (or pertaining to a class H_0). Gini poses that, in order to do so, we have to know:

* the probability P_a of obtaining x_0 (or an $x > x_0$) under H_0;

* the probability p_s that the observed x is *systematic* (i.e. produced by a distribution not belonging to H_0);

* the probability P_s that a systematic error be equal to x (or an $x > x_0$).

So, the probability π_a that a value x_0 (or an $x > x_0$) be accidental derives from a simple application of Bayes' formula:

$$\pi_a = \frac{(1-p_s)P_a}{(1-p_s)P_a + p_s P_s}\,. \tag{17}$$

Gini applied this same approach to significance testing and to interval estimation (Gini, 1943). Concerning this last topic, our aim becomes to calculate the probability that the 'true value' θ of a parameter be included within the limits a and b ($a < b$) "after having determined empirically [from a random sample] the estimate $T = t$ of this parameter". By calling I the closed interval $[a,b]$, by Bayes' formula we obtain the equality

$$\Pr(\theta \in I \mid T = t) = \frac{\Pr(\theta \in I)\Pr(T = t \mid \theta \in I)}{\Pr(\theta \in I)\Pr(T = t \mid \theta \in I) + \Pr(\theta \notin I)\Pr(T = t \mid \theta \notin I)} \tag{18}$$

It can be shown (Frosini, 2005, p. 437) that this approach is exactly equivalent to a coverage probability determined by a posterior distribution. In fact, calling I^C the complement of I with respect to the parameter space, and assuming continuous densities $f(\theta,t)$ etc., the following equalities hold:

$$P(\theta \in I)\, P(T = t \mid \theta \in I) = \int_I f(\theta,t)d\theta$$

$$P(\theta \in I^C)P(T = t \mid \theta \in I^C) = \int_{I^C} f(\theta,t)d\theta$$

$$P(\theta \in I \mid T = t) = \int_I f(\theta,t)/g(t)d\theta$$

being $g(t)$ the density of the random variable T and $f(\theta,t)/g(t)$ the (posterior) density of $\theta \mid t$.

In his criticisms of the 'accepted' theory of significance testing, Gini was rather uncompromising. Concerning e.g. the applications of formula (17), Gini explicitly poses that "if we do not possess all these pieces of information – and, in practice, *excepting particular instances, we lack them* – it is absurd claiming to be able to calculate π_a" (Gini, 1939, p. 202).

Gini is acknowledged as a forerunner of Bayesianism and, more precisely, of *objective* Bayesianism. Objectivity lies in the *existence* – and *knowledge* – of a two-stage experiment: the first stage yields a random value for the parameter θ; the second stage yields a random value of a variable X depending on a density $g(x|\theta)$.

According to Gini, either the inference is objective, both in the first and the second stage, or simply is not tenable. We could call this approach by Gini *strong* statistical inference, as a *weak* kind of statistical inference is all the same necessary.

In most applications of Bayesian inference the prior distribution chosen is simply a convenient mathematical device; in any case, a distinction must be made for the cases (*a*) when the parameter θ is not random at all, and (*b*) when a prior distribution really exists but the *chosen* prior distribution is more or less distant from the *true* one (Frosini, 2005, p.438). When the case (*a*) holds, the random variable θ degenerates in a constant $\theta = \theta_0$ with probability one. In such a case all *objective* evaluations of an interval I – hopefully including θ_0 – are of the following kind (see formula (18)):

$$\begin{aligned} P(\theta \in I \mid T = t) &= 1 \;\text{ if } \theta_0 \in I \\ &= 0 \;\text{ if } \theta_0 \in I^C \end{aligned}$$

as the degenerate θ is independent of any random variable T. Of course, if θ is the object of inference, θ_0 is unknown, and we can obtain an interval I endowed with coverage probability one only if we equate I with the whole parametric space; any experimental result reveals useless in the objective Bayesian scheme. If a researcher wishes to stick all the same to the Bayesian paradigm, all that he can do is to elicit a *subjective* evaluation of $f(\theta)$. This same approach can be followed when the available information about the prior distribution is rather loose. It is well known that another kind of interval, i.e. the *confidence interval*, can be of use in these cases.

Let us consider, for comparison purposes, a Bayesian interval of kind (17)-(18) together with a confidence interval, referred to the sampling from a normal distribution $N(\mu, \sigma^2)$ with σ^2 known. If $\bar{X}$ is the sample mean of a sample of size *n*, and $\sigma_M = \sigma/\sqrt{n}$ is its standard deviation, the inequality

$$\bar{X} - 3\sigma_M \leq \mu \leq \bar{X} + 3\sigma_M \qquad (19)$$

is clearly referred to a confidence interval for μ with coverage probability 0.9973. According to Faleschini (1947, p. 13), both statements (17)-(18) and (19) are sensible, however with a different logical content. Being $P_a = 1 - p_s$ = prior probability that μ be included in the interval (19), P_a the probability of drawing a sample with mean $\bar{X}$ from a normal random variable with μ belonging to the interval (19), p_s the probability of drawing a sample with mean $\bar{X}$ from a normal random variable with μ *not* belonging to the interval (19) (see formula (18)), we are entitled to arrive at the following interpretation: the probability 0.9973 associated with (19) is the prior probability of drawing a sample from a random variable $\mathrm{N}(\mu, \sigma^2)$ whose mean $\bar{X}$ is distant from μ less than $3\sigma_M$. Instead, the probability of type (18)-(19) is referred to the proportion of samples with mean $\bar{X}$ drawn from a normal random variable with mean μ included in the interval (19).

Of course, there is an interest to compare the (frequentist) coverage of confidence intervals and the Bayesian intervals in special cases. In any case, a confidence interval always possesses a clear probability status, while the probabilistic interpretation of an *observed* confidence interval is of the subjective kind, however objectively established (Frosini, 1996).

To make this remark clearer, let us consider a 95 per cent confidence interval *I* for parameter θ. Before the interval is determined, we say that our subjective probability – coinciding with the objective probability – that the interval obtained covers θ is 0.95; after the interval is observed, *no additional information* on the capacity of this interval of covering θ *has been gained*; thus our degree of belief that $\theta \in I$ does not change after the sample has been observed. In other words, we attach the subjective probability 0.95 to the belief that an event having objective probability 0.95 has occurred.

It can be added that this viewpoint is coherent with the usual elicitation of subjective probability, as suggested by the Bayesian school, in terms of odds of a wager: if I am willing to bet 95 dollars against 5 that a random interval includes the unknown θ, I do not see any reason to change the bet terms once the random experiment has been realised and the interval computed, since the knowledge of the experimental results does not modify the information available before the experiment was carried out (Frosini, 2009, p. 388).

Frosini (2005) has made several comparisons of frequency-based coverage in case of comparable Bayesian (assuming that a true prior distribution existed) and confidence intervals. As expected, the Bayesian coverage is dramatically lower than the nominal one when we have been too optimistic in making a too precise prior hypothesis. Anyway, it remains true that a Bayesian interval based on a very large dispersion of the prior distribution turns out to be practically equivalent to the corresponding confidence interval.

4. AN ADDENDUM TO THE TESTING OF POINT NULL HYPOTHESES

Although dealing several times with the inductive logic about the tests of significance, Gini knowingly avoided dealing with the testing of simple hypotheses in the case of parametric spaces defined on intervals or planes. Actually, the so-called *point null hypotheses* are a common device to introduce – and exemplify – the theory of testing hypotheses (Fisher-Neyman-Pearson) and the theory of parametric estimation. Quite a similar approach is followed in the applications of Bayesian theory. What is the problem, or the matter under discussion?

The problem, usually misled or neglected, is that the inductive statistical procedures should be based on observations taken from the real world, instead no exact measurement of real numbers is possible. Let us consider, for example, the hypothesis that the mean of a normal random variable N(μ, σ^2) is equal to 5 (meaning μ = 5.00…, with an infinite number of zeros): this hypothesis is certainly liable of mathematical treatment, but there is no possibility of *observing* μ = 5, or getting such a measure in the reality. Thus, we are dealing with simple hypotheses that no sample – however large – could validate. Other examples of non-observable entities are: the definition of any continuous random variable, of most parametric spaces, of the independence of two or more random variables (Frosini, 2004, p. 276). All in all, a distinction must explicitly be made between such mathematical entities (which pertain only to the ideal world of mathematics) and the possible measures (e.g. obtained through a specific measuring device). In this respect, it seems rather curious that a science, like statistics, explicitly devoted to real applications, regularly makes use of hypotheses and looks for recognising the likelihood of hypotheses using inferential procedures that cannot give the required answer.

A distressful implication of point null hypotheses results as an outcome of the *consistency property* usually required for all inference procedures (see

Frosini, 2009, p. 374). It is well known that, if we take a sufficiently large sample, any point null hypothesis is bound to be rejected. This should not surprise, as the chosen null hypothesis *cannot be true*, or at least its truthfulness cannot be evaluated. As the level of information increases, it is less and less likely to accept the null hypothesis, and this happens even if the null hypothesis – taken literally – is false or practically so (Frosini, 2004, pp. 276-277).

All this means that classical tests of (point null) hypotheses are valid just for small sample sizes, where *small* is to be deemed according to the precision of the random variable involved. When the sample size is small, the sampling variability of the test statistic is generally so large as to dominate the imprecise (being *too precise*) specification of the null hypothesis. As the sample size increases, we acknowledge the growing unsuitableness of the test procedure in order to answer the practical problem at hand. Among the possible solutions we suggest: (1) avoid applications of such tests in case of large samples, and limit the statistical analysis to estimation; (2) restate the problem in more acceptable terms, e.g. by fixing intervals for parameters.

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